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The nonlinear critical layer resulting from the spatial or temporal evolution of weakly unstable disturbances in shear flows

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A study is made of the formation of a nonlinear critical layer (first found by Benney & Bergeron 1969 and Davis 1969) in homogeneous and weakly stratified incompressible shear flows as initially small unstable disturbances develop, whose growth rate is so small that all the evolution proceeds in the 'quasi-steady' regime. It is shown that such an evolution can be described from start to finish analytically using a pair of evolution equations for the wave amplitude and phase which involve universal functions of the familiar Haberman (1972) parameter, λ, that characterizes the relative importance of the dissipation and nonlinearity.

In addition to the function of a 'logarithmic phase jump' that was introduced and investigated by Haberman (1972), the evolution equations generally also contain other functions of λ. In this paper we introduce and study (numerically and analytically) another three such functions.

1. Introduction

It is common knowledge that the critical layer problem plays the central role in shear flow stability and evolution theory at large Reynolds numbers. In the linear approximation, it manifests itself in a singularity of stationary non-dissipative hydrodynamical equations at a critical level $y = y_c$, where the wave phase velocity c equals with the flow velocity $v_x = u(y)$: $u(y_c) = c$. The reason for this is connected with the nonphysical character of the neglect of the three factors (dissipation, non-stationarity† and nonlinearity) through the entire space, including an arbitrary small neighbourhood of $y = y_c$.

Any one of these factors eliminates the singularity by modifying the equations in some neighbourhood of a critical level, $|y - y_c| < l$ (referred to as the critical layer, CL), whose scale l in each case is its own, viscous (l_v), unsteady (l_t) or nonlinear (l_N):

$$l_v = v^{1/3}, \quad l_t = \gamma \equiv |A|^{-1}\mathrm{d}|A|/\mathrm{d}s, \quad l_N = |A|^{1/2}. \tag{1.1}$$

Here v is the inverse Reynolds number, A is the complex disturbance amplitude, and s is the evolution variable (time or streamwise coordinate x); all quantities are made dimensionless by the half-thickness and the typical scale of velocity $u(y)$ variation. The CL structure is determined mainly by the factor to which the largest of the

† Non-stationarity here means the evolution of the disturbance amplitude either with time or in space (downstream).

scales (1.1) corresponds; and the others are responsible for additional details. It is therefore appropriate to distinguish the viscous, unsteady and nonlinear CL regimes.

The solution of the evolution problem of a weakly supercritical ($\gamma \ll 1$) disturbance of small amplitude is therefore constructed as if it were composed of two parts. The outer problem involves seeking the solution of a linearized non-dissipative stationary problem outside the CL, the so-called neutral mode,

$$\psi_{out} = A(s)\varphi_a(y)\,\mathrm{e}^{\mathrm{i}kx-\mathrm{i}\omega t}$$

with small corrections for the dissipation, non-stationarity and nonlinearity. At least one of these factors, however, plays the decisive role in the formation of the inner part of the solution. The parts of the solution are matched by the method of matched asymptotic expansions in the overlap domain between inner and outer regions, i.e. when $y - y_c = O(l)$, and an evolution equation of the form

$$J_1\frac{\mathrm{d}A}{\mathrm{d}s} + J_2A = \fint \mathrm{d}y\zeta_1, \tag{1.2}$$

is obtained as the condition for their compatibility. Here ζ_1 is the fundamental harmonic of the vorticity perturbation ζ inside the CL (for the time being, we will not specify the meaning of the integral $\fint$) so that the right-hand side of (1.2) is the contribution of the CL. The left-hand side, however, is fully determined by the outer solution, and J_1 and J_2 are some integrals throughout the region outside the CL. The integrands in J_1 and J_2 are quadratic in $\varphi_a(y)$, and at least one of them has a singularity (the first-order pole) when $y = y_c$, as a manifestation of the singularity of a non-dissipative stationary linear problem. The corresponding integral diverges (logarithmically), and its principal value should be taken.

In the initial stage of development of the disturbance when the amplitude is still very small and grows exponentially with a linear growth rate γ_L,

$$A = A_0\exp(\gamma_L s) \tag{1.3}$$

the nonlinear scale l_N is also small (but it does grow!), and the CL will be viscous (when $\gamma_L < \nu^{1/3}$) or unsteady (when $\gamma_L > \nu^{1/3}$). A main result of linear theory (see e.g. Drazin & Reid 1981) is that in this stage the role of the CL in both (viscous and unsteady) regimes reduces to Lin's indentation rule: the contribution of CL in (1.2) is equivalent to the indentation of a singular point $y = y_c$ in a complex plane y (from below when $u_c' > 0$ or from above when $u_c' < 0$); in this case each singular integral receives an increment proportional to the 'logarithmic phase jump'

$$\Phi_L = -\pi\,\mathrm{sgn}\,(u_c')$$

(analogous contributions arise also in the above-mentioned corrections to the neutral mode when the 'right-hand' ($y > y_c$) and the 'left-hand' ($y < y_c$) outer solutions are matched through the CL; throughout this paper it is assumed that $u_c' > 0$).

In homogeneous flows with a monotonic velocity profile $u(y)$ the neutral mode $\varphi_a(y)$ is regular (analytic) in $y = y_c$; therefore†, when supercriticality is not too small ($\gamma_L > \nu$), nonlinearity in (1.2) (which is, incidentally, formed wholly in the CL) is non-competitive until – in the course of an increase of A according (1.3) – the scale l_N becomes of the same order as the largest of l_ν and l_t, i.e. until the transition to a nonlinear CL regime (see figure 1). A nonlinear CL was discovered

† For the relationship between the neutral mode behaviour at $y = y_c$ and the nonlinear evolution character of the disturbance see Churilov & Shukhman (1992).

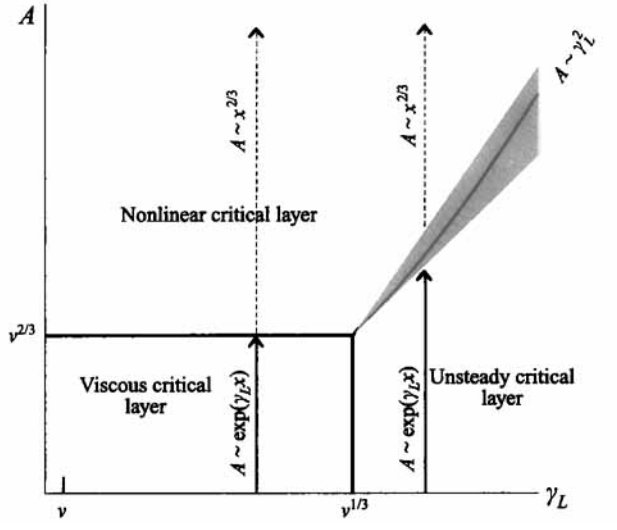

FIGURE 1. The amplitude–supercriticality diagram for a usual mixing layer.

by Benney & Bergeron (1969) and Davis (1969). They also found that in the limit $\nu/|A|^{3/2} \to 0$ matching the 'right-hand' outer solution to the 'left-hand' one through the CL requires that $\Phi_L = 0$. Subsequently, Haberman (1972) showed that in a steady problem such matching leads to a 'logarithmic phase jump'

$$\Phi_L = \Phi_1(\lambda), \qquad \lambda = \frac{\nu(u_c')^{1/2}}{k(2|A|)^{3/2}}, \tag{1.4}$$

that smoothly varies from $-\pi$ when $\lambda \to \infty$ (a viscous CL) to zero when $\lambda \to 0$ (a nonlinear CL), and calculated $\Phi_1(\lambda)$ by numerical methods. As a result, it was concluded that the role of the CL in the nonlinear evolution of unstable disturbances reduces to a new 'indentation rule' (1.4) of the singular point of the outer solution $y = y_c$. On this basis Benney & Maslowe (1975), for example, arrived at the conclusion that in the flow $u = \tanh y$ (where the principal value of J_1 is equal to zero) the evolution equation in the limit $\lambda \ll 1$ must be of the second order in s (rather than of the first order as (1.2)).

Subsequent investigations demonstrated, however, that the role of the CL in the formation of the evolution equation (1.2) and hence the evolution character of unstable disturbances is far more complicated and significant. Specifically, it was shown that nonlinear problems do not and cannot have any 'indentation rule'. That gave impetus to a new series of publications on the study of the evolution with the transition to the nonlinear CL regime.

The transition itself was studied most extensively and specifically by Goldstein & Hultgren (1988) who derived for the spatial evolution problem an equation describing the dynamics of vorticity ζ inside the CL and solved it simultaneously with (1.2) numerically at different values of the parameter $\sigma = \nu/\gamma_L^3$. Their calculations showed that far from the boundary of the nonlinear CL region on the amplitude–supercriticality diagram (figure 1) the amplitude always behaves qualitatively as follows from the 'phase jump' concept, i.e. it grows exponentially at a small amplitude, and varies as (Huerre & Scott 1980)

$$A \sim (\gamma_L \nu s)^{2/3} \tag{1.5}$$

at a large amplitude. However, a wide variety of transitions ranging from a monotonic transition when $\sigma \gg 1$ to a strongly oscillating one when $\sigma \ll 1$, are observed between these two asymptotic representations. The reason for this is readily understood from simple physical considerations.

The disturbance alters the flow topology: instead of straight streamlines parallel to the x-axis, there appears a cat's-eye configuration, with fluid particles trapped inside. The distribution of vorticity ζ inside the CL is now determined by three processes: (i) a growth in amplitude accompanied by the capture of an increasingly growing number of particles (the typical time† $\tau_e = \gamma^{-1}$), (ii) the diffusion of ζ (the typical time $\tau_d = L^2/\nu$, L being the scale of variation of ζ), and (iii) the motion and mixing of particles inside the cat's eye (the typical time $\tau_m = A^{-1/2}$). In the process of evolution the relationships between τ_e, τ_d and τ_m can change drastically, which adds great complexity to the picture of development of the disturbance.

In the initial (linear) stage L coincides with the CL scale l and $\tau_m \gg \max(\tau_e, \tau_d)$, i.e. trapped particles are practically at rest, and the vorticity distribution inside the CL hardly differs from the undisturbed one. The transition to a nonlinear CL regime starts when the amplitude has increased to the extent that τ_m becomes of the order of $\min(\tau_d, \tau_e)$ and a vorticity redistribution inside the cat's eyes will set in as a consequence of the motion of fluid particles.

When the transition begins from the unsteady CL regime ($\sigma \ll 1$) the viscosity is small, and the vorticity ζ is frozen-in to the flow and is transported by fluid particles. Therefore, in about every half-rotation of trapped particles there is a change of sign of the vorticity inside the cat's eyes, and along with it the sign of the right-hand side of (1.2) – the amplitude begins to oscillate with a period of order τ_m. Ultimately, because of the non-isochronicity of neighbouring orbits fluid particles initially nearby will move away from each other and a mixing of trapped particles will occur. In view of the fact that each fluid particle carries its own value of the vorticity, the distribution of ζ will become fine-scaled as the result of this mixing. With the scale of refinement $L \sim \sigma^{1/2} l_N$, the diffusion time τ_d will become equal to τ_m, the viscosity will smooth out the distribution of ζ, and inside the cat's eyes there will appear a plateau on the ζ-profile, slightly sloping in the same direction as outside the cat's eyes. As a result, the growth rate of amplitude A decreases sharply, the viscosity ensures a smooth distribution of ζ inside the CL and its fast adjustment to the local instataneous value of A, and the evolution law (1.5) is established.

When the transition begins from the viscous CL regime ($\sigma \gg 1$) the evolution of ζ is quasi-steady from the outset: the viscosity is sufficient to ensure a smooth distribution of ζ and its adjustment to A. This greatly simplifies the problem because ζ can be calculated independently if A is assumed to be given, and the evolution equation (1.2) can be obtained in a closed form suitable for a description of the development of a disturbance in all stages, from (1.3) to (1.5).

This paper is concerned with the study of such a 'quasi-steady' evolution of unstable disturbances in shear flows of homogeneous and weakly stratified fluid and primarily with the derivation of appropriate evolution equations. These equations are the same in form with those obtained by means of the 'indentation rule'. But if they are treated from such a point of view, it turns out that the 'rules' are different not only for different problems and not only for the integrals J_1 and J_2 of the same problem, but even for the real and imaginary parts of the same integral. In addition to the Haberman function $\Phi_1(\lambda)$, we introduce and calculate three further functions: $\Phi_2(\lambda)$,

† To be more precise, the typical scale on the scale of the evolution variable s.

which is the second hypostasis of $\Phi_1(\lambda)$ because they co-exist in all non-degenerate problems; $\Phi_3(\lambda)$ related to an interesting stabilization phenomenon of the instability due to a finiteness of the instability region width on the disturbed vorticity profile; and $\Phi_4(\lambda, Pr)$ that represents the influence of a weak stratification (Pr being the Prandtl number).

Resulting equations can be solved by quadratures and this makes possible an analytic study of the development of unstable disturbances at all evolution stages from (1.3) to a saturation (if any) or to the final stage of the form (1.5). In addition to being of independent interest, such solutions are of value as tests for numerical schemes designed for solving more complicated unsteady problems occurring in the case of a large supercriticality.

The subsequent presentation is organized as follows. Section 2 gives an outline of the derivation of equations both for the temporal and spatial evolution of shear flows of homogeneous incompressible fluid which takes into account (for completeness of our treatment) the β-effect. We use the results to consider temporal evolution problems for disturbances in a weakly supercritical flow on the β-plane and temporal and spatial evolution problems for weakly supercritical disturbances in a usual mixing layer. In this connection we introduce and calculate the function $\Phi_2(\lambda)$.

Section 3 considers the stabilization of disturbances in a weakly supercritical flow on the β-plane in the nonlinear CL regime due to a finite width of the instability region on the generalized vorticity profile, and the function $\Phi_3(\lambda)$ is also introduced and studied, which is related to a description of this process.

In §4 we obtain evolution equations for a slightly stratified flow and study the function $\Phi_4(\lambda, Pr)$ which appears in them. All regimes and stages of evolution are considered in detail.

Section 5 discusses results obtained and their implications.

In Appendix A we calculate asymptotic representations of the previously introduced functions in the limit of a nonlinear CL ($\lambda \ll 1$) with a detailed treatment of a narrow region of width $O(\lambda^{1/2})$ near the cat's-eye boundary, as done by Brown & Stewartson (1978) for $\Phi_1(\lambda)$. In Appendix B we describe a numerical computation algorithm for the Φ_i.

2. Quasi-steady evolution of disturbances in a homogeneous shear flow (the functions $\Phi_1(\lambda)$ and $\Phi_2(\lambda)$)

Consider a homogeneous shear flow with a monotonically increasing velocity profile, $u'(y) > 0$. It is known that the phase velocity $c_0 = \omega_0/k_0$ of a neutral mode (with frequency ω_0 and wavenumber k_0) is such that at a critical level $y = y_c$ a generalized vorticity has an extremum. This means

$$u''(y_c) = 0 \tag{2.1}$$

for a usual mixing layer, and

$$u''(y_c) - \beta = 0 \tag{2.2}$$

for a zonal flow on the β-plane.

Let a perturbation of the stream function be represented as

$$\delta\psi = \left(A e^{ik_0(x-ct)} + \text{c.c.}\right) \varphi_a(y) + \ldots \tag{2.3}$$

where $\varphi_a(y)$ is an eigenfunction of the neutral mode (real, in view of the equality (2.1) or (2.2) and normalized by the condition $\varphi_a(y_c) = 1$). Dots denote terms of higher order of smallness.

A weakly unstable disturbance is differentiated from the neutral mode by its parameters ($k \neq k_0$ and/or $\omega \neq \omega_0$), and its amplitude A evolves slowly with time or in space – accordingly, the temporal or spatial evolution is said to occur.

In temporal-evolution problems for flows with a spectrum of unstable disturbances wide in k the desired weakly supercritical mode is specified by the choice $k = k_0 + \Delta k$ ($\Delta k < 0$)†; however, in the case of a weakly supercritical flow with a narrow spectrum of unstable modes where the controlling parameter is slightly less than its critical value, it is natural to study the mode $k_0 = k_{cr}$, i.e. the most unstable mode. In particular, in the case when the β-effect is taken into account, the neutral curve on the (k, β)-plane has a maximum

$$\beta_{max} = \beta_{cr} = \max u''(y) \tag{2.4}$$

at $k = k_{cr}$ and, according to (2.2), the CL is the vicinity of a point where

$$u_c''' \equiv u'''(y_c) = 0 \ . \tag{2.5}$$

The spatial-evolution problem is usually posed in flows with a spectrum of unstable modes wide in ω, and the desired mode is separated through excitation at the frequency $\omega = \omega_0 + \Delta\omega$ ($\Delta\omega < 0$) and develops streamwise (for a detailed justification to such a setting out of the problem see Goldstein & Hultgren 1988). We will consider examples covering all of the cases described above.

The technique for deriving evolution equations is well known; therefore, we will only introduce the notation and scaling. We introduce the parameter $\varepsilon \ll 1$ that characterizes the disturbance amplitude, and put

$$A = \varepsilon B(\xi, \tau) \exp[\mathrm{i}\Theta(\xi, \tau)], \tag{2.6}$$

$$\nu = \eta\varepsilon^{3/2} \ . \tag{2.7}$$

Here

$$\tau = \mu\varepsilon^{1/2}t, \quad \xi = \mu\varepsilon^{1/2}x, \tag{2.8}$$

which corresponds to the following scaling of the quantities Δk, $\Delta\omega$ and $\Delta\beta$:

$$\Delta k = \mu\varepsilon^{1/2}K, \quad \Delta\omega = \mu\varepsilon^{1/2}\Omega, \quad \Delta\beta = \beta - \beta_{cr} = \mu\varepsilon^{1/2}\beta_1, \quad \beta_1 < 0. \tag{2.9}$$

The amplitude εB and the phase Θ of the wave depend on the slow (evolution) variables τ and ξ. The quantities K, Ω and β_1 are of the order of unity. In general η and μ are also of the order of unity, which means that we are working in the vicinity of a 'triple point' $\gamma_L = \nu^{1/3}$, $A = \nu^{2/3}$ (see figure 1) where the three possible CL regimes are concurrent. However, by varying η and μ, it is possible to extend significantly the 'working zone' boundaries both in γ_L and in A and cover, in the framework of the same equations, all possible types of evolution with the transition to a nonlinear CL regime. As pointed out in the Introduction (and is evident from figure 1), such a transition from the viscous CL regime ($\gamma_L < \nu^{1/3}$) occurs when $A \sim \nu^{2/3} \gg \gamma_L^2$, which corresponds to $\eta = O(1)$ and $\mu \ll 1$ and implies, as will be shown below, the quasi-steady development of vorticity ζ inside the CL. On the other hand, the transition from the unsteady CL regime ($\gamma_L \gg \nu^{1/3}$) proceeds when $A \sim \gamma_L^2 \gg \nu^{1/3}$,

† It is clear that such a setting out of the problem is somewhat artificial: there exist more unstable modes and their development will disturb the picture obtained; we consider this case only as an instructive aid.

which requires $\eta \ll 1$ and $\mu = 1$; the evolution of ζ in this case will be unsteady, as described in Introduction.

By representing the stream function ψ inside the CL as

$$\psi = \varepsilon^{1/2} c Y + (\varepsilon \Psi^{(1)} + \varepsilon^{3/2} \Psi^{(3/2)} + \varepsilon^2 \Psi^{(2)} + \ldots), \quad Y = (y - y_c)/\varepsilon^{1/2},$$

we obtain, as a result of matching to $O(\varepsilon^2)$ of the inner and outer solutions:

$$\mu \left(\frac{\partial}{\partial \tau} + c \frac{\partial}{\partial \xi} \right) \zeta + k_0 u_c' [Y - Y_c(\xi, \tau)] \frac{\partial \zeta}{\partial \theta} + 2 k_0 B \sin\theta \frac{\partial \zeta}{\partial Y} - \eta \frac{\partial^2 \zeta}{\partial Y^2}$$
$$= -2 \frac{u_c'''}{u_c'} \cos\theta \left(\frac{\partial}{\partial \tau} + c \frac{\partial}{\partial \xi} \right) B + 2B \sin\theta \left[\frac{u_c'''}{u_c'} \left(\frac{\partial}{\partial \tau} + c \frac{\partial}{\partial \xi} \right) \Theta + k_0 \beta_1 \right], \quad (2.10)$$

$$I_1 \left(\frac{\partial}{\partial \tau} + c \frac{\partial}{\partial \xi} \right) B - 2 k_0^2 I_2 \frac{\partial B}{\partial \xi} = k_0 \fint_{-\infty}^{\infty} \mathrm{d}Y \langle \zeta \sin\theta \rangle, \quad (2.11)$$

$$B I_1 \left(\frac{\partial}{\partial \tau} + c \frac{\partial}{\partial \xi} \right) \Theta - 2 k_0^2 B I_2 \frac{\partial \Theta}{\partial \xi} + \beta_1 k_0 B I_3 = k_0 \fint_{-\infty}^{\infty} \mathrm{d}Y \langle \zeta \cos\theta \rangle, \quad (2.12)$$

where

$$\zeta = \mu^{-1} \left[\frac{\partial^2}{\partial Y^2} \Psi^{(2)} - \tfrac{1}{2} u_c''' Y^2 - 2 \left(\frac{u_c'''}{u_c'} + k_0^2 \right) B \cos\theta \right],$$
$$\theta = k_0 (x - ct) + \Theta(\xi, \tau), \quad (2.13)$$

$$Y_c(\xi, \tau) = -\frac{\mu}{k_0 u_c'} \left(\frac{\partial}{\partial \tau} + c \frac{\partial}{\partial \xi} \right) \Theta \; . \quad (2.14)$$

Equations (2.11) and (2.12) are nothing more nor less than equation (1.2), divided into the real and imaginary parts and slightly transformed, while (2.10) describes the vorticity dynamics inside the CL. It is convenient to formulate relevant boundary conditions when $Y \to \pm\infty$ as

$$\frac{\partial \zeta}{\partial Y} \to 0 \, . \quad (2.15)$$

In (2.11) and (2.12)

$$\fint_{-\infty}^{\infty} \mathrm{d}Y (\cdots) = \lim_{Z \to \infty} \int_{-Z}^{Z} \mathrm{d}Y (\cdots), \qquad \langle \cdots \rangle = \frac{1}{2\pi} \int_0^{2\pi} \mathrm{d}\theta (\cdots),$$

$$I_1 = \fint_{-\infty}^{\infty} \mathrm{d}y \varphi_a^2(y) \frac{u'' - \beta_{cr}}{(u - c)^2}, \quad I_2 = \int_{-\infty}^{\infty} \mathrm{d}y \varphi_a^2(y), \quad I_3 = \fint_{-\infty}^{\infty} \mathrm{d}y \frac{\varphi_a^2(y)}{u - c}, \quad (2.16)$$

where $\fint \mathrm{d}y(\ldots)$ stands for the half the sum of corresponding integrals taken with indentation of the singular point from above and below.

As pointed out in the above discussion, for transitions of interest here, from the viscous CL regime we have $\mu \ll 1$, and the first (evolution) term in (2.10) is small, and this implies a quasi-steady adjustment of ζ to the local instantaneous amplitude B throughout the duration of the disturbance evolution in accordance with the equation

$$k_0 u_c' Y \frac{\partial \zeta}{\partial \theta} + 2 k_0 B \sin\theta \frac{\partial \zeta}{\partial Y} - \eta \frac{\partial^2 \zeta}{\partial Y^2}$$
$$= -2 \frac{u_c'''}{u_c'} \cos\theta \left(\frac{\partial}{\partial \tau} + c \frac{\partial}{\partial \xi} \right) B + 2B \sin\theta \left[\frac{u_c'''}{u_c'} \left(\frac{\partial}{\partial \tau} + c \frac{\partial}{\partial \xi} \right) \Theta + k_0 \beta_1 \right] \; . \quad (2.17)$$

To evaluate the integrals

$$\int \mathrm{d}Y \langle \zeta \sin\theta \rangle \quad \text{and} \quad \int \mathrm{d}Y \langle \zeta \cos\theta \rangle, \tag{2.18}$$

appearing on the right-hand sides of (2.11) and (2.12), we introduce auxiliary functions $g_1(\lambda; z, \theta)$ and $g_2(\lambda; z, \theta)$, such that

$$\left(-\lambda\frac{\partial^2}{\partial z^2} + z\frac{\partial}{\partial\theta} + \sin\theta\frac{\partial}{\partial z}\right)\begin{pmatrix} g_1 \\ g_2 \end{pmatrix} = -2\begin{pmatrix} \sin\theta \\ \cos\theta \end{pmatrix}, \tag{2.19}$$

$$\frac{\partial g_{1,2}}{\partial z} \to 0 \text{ as } z \to \pm\infty$$

and using them we define two functions of λ:

$$\Phi_1(\lambda) = \int_{-\infty}^{\infty} \mathrm{d}z \langle g_1 \sin\theta \rangle, \quad \Phi_2(\lambda) = \int_{-\infty}^{\infty} \mathrm{d}z \langle g_2 \cos\theta \rangle. \tag{2.20}$$

Note that the symmetry properties of (2.19) yield

$$\int_{-\infty}^{\infty} \mathrm{d}z \langle g_1 \cos\theta \rangle = \int_{-\infty}^{\infty} \mathrm{d}z \langle g_2 \sin\theta \rangle = 0\,.$$

The plots of $\Phi_1(\lambda)$ and $\Phi_2(\lambda)$ are shown in figure 2. Thin lines correspond to their asymptotic expansions at small and large λ:

$$\Phi_1(\lambda) = \begin{cases} C^{(1)}\lambda + D^{(1)}\lambda^{3/2} + O(\lambda^2), & \lambda \ll 1 \\ -\pi + a_1\lambda^{-4/3} + O(\lambda^{-8/3}), & \lambda \gg 1 \end{cases} \tag{2.21}$$

$$\Phi_2(\lambda) = \begin{cases} C^{(2)}\lambda^{-1} + O(1), & \lambda \ll 1 \\ -\pi + a_2\lambda^{-4/3} + O(\lambda^{-8/3}), & \lambda \gg 1 \end{cases} \tag{2.22}$$

Here

$$\left.\begin{aligned} &C^{(1)} = -5.5151\ldots, \quad D^{(1)} = 4.2876\ldots, \quad a_1 = 1.6057\ldots, \\ &C^{(2)} = -2.5008\ldots, \quad a_2 = -a_1. \end{aligned}\right\} \tag{2.23}$$

The function $\Phi_1(\lambda)$ was introduced by Haberman (1972), and he also calculated its asymptotic representations† and presented, based on numerial calculations, its plot. The function $\Phi_2(\lambda)$ was introduced by Shukhman (1989) and he also calculated its asymptotic representations (2.22)‡. A numerical calculation of Φ_2 throughout the range of variation of λ is done for the first time here.

The integrals of (2.18) are readily evaluated in terms of Φ_1 and Φ_2; as a result, from (2.11) and (2.12) we obtain a pair of evolution equations for the amplitude and phase of the wave:

$$I_1\left(\frac{\partial}{\partial\tau} + c\frac{\partial}{\partial\xi}\right)B - 2k_0^2 I_2\frac{\partial B}{\partial\xi} = -\Phi_1(\lambda(\xi,\tau))\left[\frac{u_c'''}{u_c'^2}\left(\frac{\partial}{\partial\tau} + c\frac{\partial}{\partial\xi}\right)\Theta + \frac{k_0\beta_1}{u_c'}\right]B, \tag{2.24}$$

† The term $D^{(1)}\lambda^{3/2}$ in (2.21) seems to be given for the first time, although all pertinent calculations were reported by Brown & Stewartson (1978).

‡ An asymptotic representation of Φ_2 as $\lambda \ll 1$ was also obtained by Goldstein & Hultgren (1988) but with the incorrect value of $C^{(2)} = -2.7214\ldots$

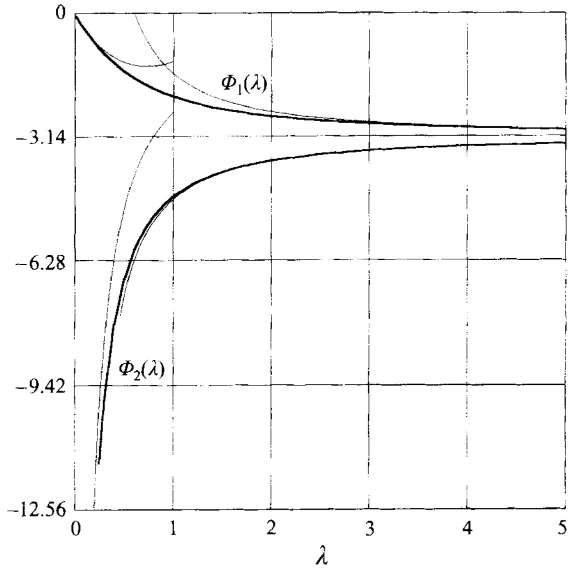

FIGURE 2. The functions $\Phi_1(\lambda)$ and $\Phi_2(\lambda)$. Thin lines show their asymptotic expansions (2.21) and (2.22).

$$\left[I_1\left(\frac{\partial}{\partial\tau}+c\frac{\partial}{\partial\xi}\right)\Theta-2k_0^2I_2\frac{\partial\Theta}{\partial\xi}+\beta_1k_0I_3\right]B=\Phi_2\left(\lambda(\xi,\tau)\right)\frac{u_c'''}{u_c'^2}\left(\frac{\partial}{\partial\tau}+c\frac{\partial}{\partial\xi}\right)B, \quad (2.25)$$

$$\lambda(\xi,\tau)\equiv\frac{\eta(u_c')^{1/2}}{k_0\left(2B(\xi,\tau)\right)^{3/2}}\cdot \quad (2.26)$$

It is interesting to look at the equations obtained from the point of view of the 'indentation rule'. The left-hand sides of (2.24) and (2.25) involve 'finite parts' of two singular (logarithmically divergent) integrals I_1 and I_3 which at $y=y_c$ have the residues

$$\mathrm{Res}_1=\frac{u_c'''}{u_c'^2}\qquad \mathrm{Res}_3=\frac{1}{u_c'}$$

respectively. Taking into consideration that the complex amplitude $A=\varepsilon B\exp(\mathrm{i}\Theta)$, one can to attribute to the integral I_3 with the indentation of the point $y=y_c$ the 'logarithmic phase jump' $\Phi_1(\lambda)$; however, two different 'jumps', $\Phi_1(\lambda)$ and $\Phi_2(\lambda)$, have to be simultaneously attributed to the integral I_1, and one of them (Φ_2) tends to infinity in the limit of a fully developed nonlinear CL ($\lambda\to 0$)!

In what follows we will consider several examples in which the evolution equations contain the functions $\Phi_1(\lambda)$ and $\Phi_2(\lambda)$ in different combinations.

2.1. *Weakly supercritical zonal flow $u=\tanh y$ on the β-plane; temporal evolution*

In this problem the spectrum of unstable modes is narrow, and one should put $k_0=k_{cr}$ and, accordingly, $\partial B/\partial\xi=\partial\Theta/\partial\xi=0$, while the supercriticality is due to $\Delta\beta$. Parameters, the eigenfunction φ_a of the neutral mode and the integrals I_1 and I_3

are thus (Churilov & Shukhman 1987)

$$c = -(1/3)^{1/2}, \quad k_0 = (1-c^2)^{1/2} = (2/3)^{1/2},$$

$$\beta_{cr} = u_c'' = -2c(1-c^2) = 4/3^{3/2}, \quad u_c' = 2/3, \quad u_c''' = 0, \quad u_c^{\text{iv}} = -16/3^{3/2},$$

$$\varphi_a(y) = (1-\tanh y)^{(1+c)/2}(1+\tanh y)^{(1-c)/2}/\varphi_c, \quad \varphi_c = (1-c)^{(1+c)/2}(1+c)^{(1-c)/2},$$

$$I_1 = \frac{4\pi c^2}{\sin(\pi c)}\varphi_c^{-2} < 0, \quad I_3 = -\frac{\pi}{\sin(\pi c)}\left[1 - \left(\frac{1-c}{1+c}\right)^c \cos(\pi c)\right]\varphi_c^{-2}\,. \tag{2.27}$$

As a result, from (2.24) and (2.25) we obtain

$$\frac{\mathrm{d}B}{\mathrm{d}\tau} = \gamma_* \left(\frac{\Phi_1(\lambda)}{-\pi}\right) B, \quad \lambda = \eta(2B)^{-3/2}, \tag{2.28}$$

$$\frac{\mathrm{d}\Theta}{\mathrm{d}\tau} = -c_1 k_0. \tag{2.29}$$

Here

$$\gamma_* = \frac{\pi k_0 \beta_1}{u_c' I_1} = \frac{k_0\beta_1}{4u_c' c^2}\sin(\pi c)\varphi_c^2 \tag{2.30}$$

is the growth rate of linear theory, and

$$c_1 = \beta_1 \frac{I_3}{I_1} = -\frac{\beta_1}{4c^2}\left[1 - \left(\frac{1-c}{1+c}\right)^c \cos(\pi c)\right]$$

is a correction to the phase velocity of the wave.

The frequency shift, defined by (2.29), is due to the supercriticality and does not depend on the amplitude. This means that the CL position remains unchanged as the amplitude grows. Equation (2.28) is readily integrable by quadratures and provides a means for following the entire evolution process, starting from a linear stage ($\lambda \gg 1$) and ending with the evolution in the nonlinear CL regime ($\lambda \ll 1$) when a reduction in growth rate $\gamma_* \rightarrow \gamma_*\lambda$ leads to a power-law evolution:

$$B \propto (\eta\gamma_*\tau)^{2/3}\,. \tag{2.31}$$

2.2. *Mixing layer* $u = \tanh y$; *temporal evolution*

In this problem there is a wide spectrum of unstable modes and the supercriticality is due to Δk. In (2.24), (2.25) we put

$$c = 0, \quad k_0 = 1, \quad u_c' = 1, \quad u_c'' = 0, \quad u_c''' = -2, \quad \varphi_a(y) = 1/\cosh(y), \quad I_1 = 0, \quad I_2 = 2,$$

$$\frac{\partial\Theta}{\partial\xi} = K, \quad \frac{\partial B}{\partial\xi} = 0\,.$$

We obtain, respectively,

$$\frac{\mathrm{d}\Theta}{\mathrm{d}\tau}\Phi_1 = 0,$$

$$4KB = 2\frac{\mathrm{d}B}{\mathrm{d}\tau}\Phi_2,$$

whence it follows that the phase velocity of the wave, as in the preceding case, is not influenced by nonlinearity, and the amplitude obeys the equation

$$\frac{\mathrm{d}B}{\mathrm{d}\tau} = \gamma_* \left(\frac{-\pi}{\Phi_2(\lambda)} \right) B, \quad \lambda = \eta (2B)^{-3/2}, \tag{2.32}$$

where

$$\gamma_* = -\frac{2K}{\pi}$$

is a linear growth rate. The comparison of (2.32) with (2.28) reveals that their solutions behave in qualitatively the same manner because, as can be seen from the asymptotic expansions (2.21) and (2.22) and from figure 2, the functions $\Phi_1(\lambda)$ and $\pi^2/\Phi_2(\lambda)$ have similar behaviour. Nevertheless, these are different functions as they have quite a different origin.

This problem was solved by Huerre & Scott (1980) who obtained an equation of the form (2.32) but with Φ_1 instead of Φ_2†.

The problems considered above are in a sense degenerate. In the first example the condition $u_c''' = 0$ leads to the fact that the evolution equation (2.28) contains a 'sine' function $\Phi_1(\lambda)$ alone, while in the second example, because of the flow antisymmetry, $I_1 = 0$ and (2.32) involves only a 'cosine' function $\Phi_2(\lambda)$. Consider next the non-degenerate case.

2.3. *Mixing layer with an arbitrary velocity profile; temporal evolution*

In this problem $\beta_1 = 0$, $u_c'' = 0$, $\partial\Theta/\partial\xi = K$, $\partial B/\partial\xi = 0$. From (2.24) and (2.25), with a little manipulation, we get

$$\frac{\mathrm{d}B}{\mathrm{d}\tau} = -2k_0^2 K \frac{u_c'''}{u_c'^2} \frac{I_2 \Phi_1(\lambda)}{I_1^2 + \Phi_1(\lambda)\Phi_2(\lambda)(u_c'''/u_c'^2)^2} B, \tag{2.33}$$

$$\frac{\mathrm{d}\Theta}{\mathrm{d}\tau} = 2k_0^2 K \frac{I_1 I_2}{I_1^2 + \Phi_1(\lambda)\Phi_2(\lambda)(u_c'''/u_c'^2)^2}, \tag{2.34}$$

$$\lambda(\tau) \equiv \frac{\eta (u_c')^{1/2}}{k_0 (2B(\tau))^{3/2}} .$$

Here, unlike the cases considered above, the correction to the frequency is amplitude-dependent; hence, as the evolution proceeds, there is a change in the phase velocity of the wave and in the position of a critical level $y_c = y_c(t)$. From (2.14), via (2.21), (2.22) and (2.34), we obtain the total displacement of the critical level in the course of the evolution:

$$\Delta y_c = -\frac{2k_0 \Delta k}{u_c'} I_1 I_2 \left\{ \frac{1}{I_1^2 + C^{(1)}C^{(2)}(u_c'''/u_c'^2)^2} - \frac{1}{I_1^2 + \pi^2 (u_c'''/u_c'^2)^2} \right\} . \tag{2.35}$$

The amplitude growth, described by (2.33), does not differ qualitatively from previous cases. Indeed, as is evident from figure 3, the product $\Phi_1\Phi_2$ does not change too greatly with a change of λ from 0 to ∞. Neglecting this change one may write (2.33) approximately in form of (2.28) with a linear growth rate:

$$\gamma_* = 2\pi k_0^2 K \frac{u_c'''}{u_c'^2} \frac{I_2}{I_1^2 + \pi^2 (u_c'''/u_c'^2)^2} .$$

† A correct result can be obtained if, in addition to a jump of the coefficient of $\exp(\mathrm{i}\Theta)$, a similar jump for $\exp(-\mathrm{i}\Theta)$ is introduced into equation (3.9) of Huerre & Scott (1980).

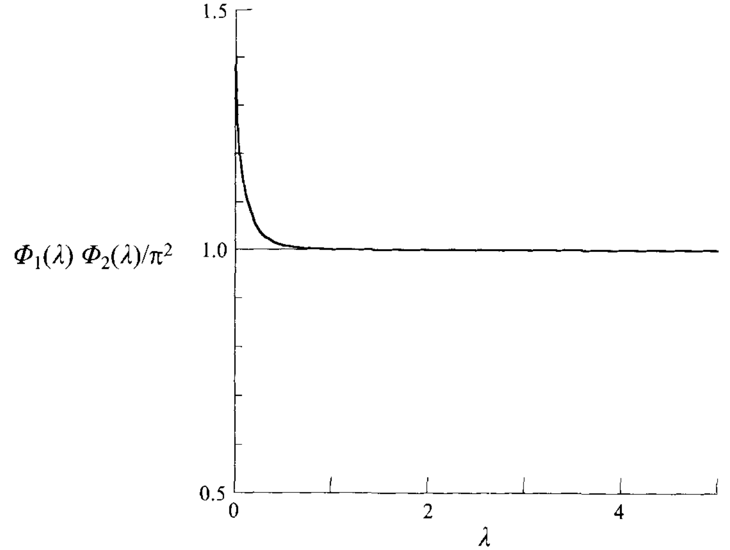

FIGURE 3. The product $(-\Phi_1/\pi) \times (-\Phi_2/\pi)$ *vs.* λ.

2.4. *Mixing layer with an arbitrary velocity profile; spatial evolution*

In this problem $\partial B/\partial \tau = 0$, $\partial \Theta/\partial \tau = -\Omega$ $(\Omega < 0)$ and equations (2.24), (2.25) yield

$$\frac{\mathrm{d}B}{\mathrm{d}\xi} = -\frac{2k_0^2}{c^2}\frac{u_c'''}{u_c'^2}\frac{\Phi_1(\lambda)}{H(\lambda)} I_2 \Omega B, \tag{2.36}$$

$$\frac{\mathrm{d}\Theta}{\mathrm{d}\xi} = \frac{\Omega}{c}\left[1 + 2k_0^2 \frac{I_0 I_2}{cH(\lambda)}\right], \tag{2.37}$$

$$H(\lambda) = I_0^2 + \Phi_1(\lambda)\Phi_2(\lambda)(u_c'''/u_c'^2)^2, \quad I_0 \equiv I_1 - 2k_0^2 I_2/c\,. \tag{2.38}$$

The qualitative behaviour of the amplitude and phase is the same as in the preceding example. Note that in the case of the spatial evolution the equations contain – with any velocity profile – both functions Φ_1 and Φ_2.

The evolution equations (2.28), (2.32), (2.33) and (2.36) obtained above have the form

$$\frac{\mathrm{d}B}{\mathrm{d}s} = \gamma_* R(\lambda) B, \tag{2.39}$$

where s is an evolution variable (τ or ξ), and $R(\lambda)$ is a reducing factor which is expressed differently in each case in terms of the functions $\Phi_1(\lambda)$ and $\Phi_2(\lambda)$. At the transition to a nonlinear CL regime the growth rate is reduced, the rate at which the amplitude increases drops abruptly, and the evolution becomes of power-law type (1.5) (see also (2.31)).

At this stage it is sufficient to have a small additional effect to stop a growth of B and even cause the wave to be damped. One main such effect, viscous broadening of the flow profile, was considered by Goldstein & Hultgren (1988) and Hultgren (1992) (see also Churilov & Shukhman 1994). Broadening leads to a linear (in a first approximation) in s decrease of γ_* in (2.39):

$$\gamma_* \to \gamma_* - a\eta s$$

and, in a time $s \sim \gamma_*/\eta$, not only to growth stoppage but also to the dissipation of the wave.

In some cases, however, the flow is produced and sustained by an external force f, and its velocity profile is the result of counterbalancing f by forces of viscous friction: $f = -\nu\Delta^2\psi_{00}$ where ψ_{00} is a stream function of an unperturbed flow. Such a flow does not broaden, and some different, weaker stabilization mechanisms are at work here. One of them, which is associated with the smallness of the unstable region width in weakly supercrtical flows, was treated by Churilov (1989) and Shukhman (1989) in a zonal flow on the β-plane and in a differentially rotating fluid, respectively. New terms in the nonlinear evolution equation and a new function $\Phi_3(\lambda)$ are related to this mechanism.

3. Stabilization of a zonal flow on the β-plane (the function $\Phi_3(\lambda)$)

In a zonal flow, the force f is represented by the pressure gradient. The instability is due to the fact that when $\beta < \beta_{cr} = \max u''(y) \equiv u_c''$ a generalized vorticity has a positive derivative, ($u_c'' - \beta > 0$) in the vicinity of $y = y_c$. The width L of this region is determined from the equation (it will be recalled that $u_c''' = 0$ and, obviously, $u_c^{\text{iv}} < 0$):

$$u''(y) - \beta = u_c'' - \beta + \tfrac{1}{2}u_c^{\text{iv}}(y-y_c)^2 = -\Delta\beta + \tfrac{1}{2}u_c^{\text{iv}}(y-y_c)^2 = 0$$

and is

$$L = \left(\frac{2\Delta\beta}{u_c^{\text{iv}}}\right)^{1/2} = O(\gamma_L^{1/2}) .$$

The instability will be stabilized when this region is totally embedded in a nonlinear CL (of width $l_N = |A|^{1/2}$), and because of mixing of its constituent liquid particles $u_c'' - \beta \approx 0$ will be established. This will occur when the amplitude

$$A = O(\gamma_L) \tag{3.1}$$

in a time (see (2.31)) $t \sim \gamma_L^{1/2}/\nu$.

As shown by Churilov (1989), taking this mechanism into account modifies the evolution equation (2.28) thus:

$$\frac{\mathrm{d}B}{\mathrm{d}\tau} = -\frac{k_0\sin(\pi c)}{4\pi u_c' c^2}\varphi_c^2\left[\beta_1 B\Phi_1(\lambda) + \frac{u_c^{\text{iv}}}{8u_c'}B^2\Phi_3(\lambda)\right] . \tag{3.2}$$

Here

$$\Phi_3(\lambda) = \fint_{-\infty}^{\infty}\mathrm{d}z\langle g_3\sin\theta\rangle, \tag{3.3}$$

and the function $g_3(\lambda; z, \theta)$ is the solution of the equation

$$\left(-\lambda\frac{\partial^2}{\partial z^2} + z\frac{\partial}{\partial\theta} + \sin\theta\frac{\partial}{\partial z}\right)g_3 = 8\sin 2\theta \tag{3.4}$$

with the boundary conditions

$$\frac{\partial g_3}{\partial z} \to 0 \ \text{ as } \ z \to \pm\infty . \tag{3.5}$$

The function $\Phi_3(\lambda)$ is shown in figure 4. In the limits $\lambda \ll 1$ and $\lambda \gg 1$ it was calculated

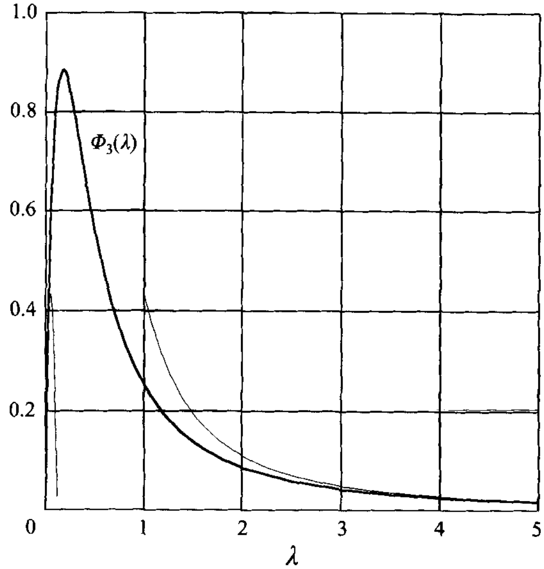

FIGURE 4. The function $\Phi_3(\lambda)$ and its asymptotic expansions (3.6) (thin lines).

analytically by Churilov (1989) and Shukhman (1989) (without the term $D^{(3)}\lambda^{3/2}$)):

$$\Phi_3(\lambda) = \begin{cases} C^{(3)}\lambda + D^{(3)}\lambda^{3/2} + O(\lambda^2), & \lambda \ll 1, \\ a_3\lambda^{-2} + O(\lambda^{-10/3}), & \lambda \gg 1, \end{cases} \tag{3.6}$$

$$C^{(3)} = 24.012\ldots, \quad D^{(3)} = -16D^{(1)} = -68.60\ldots, \quad a_3 = 0.4389\ldots$$

The right-hand side of (3.2) goes to zero with the saturation amplitude $B = B_{sat}$, defined by the relationship (values of the parameters are given in (2.27)):

$$\frac{\Phi_3(\lambda_{sat})}{\lambda_{sat}^{2/3}\Phi_1(\lambda_{sat})} = -\frac{16\beta_1}{u_c^{\text{iv}}}\left(\frac{k_0 u_c'}{\eta}\right)^{2/3} = \frac{2\beta_1}{(3\eta^2)^{1/3}}; \quad \lambda_{sat} = \eta(2B_{sat})^{-3/2}.$$

The saturation amplitude B_{sat} depends on the supercriticality, and from (3.6) and (2.21) it follows that when $|\beta_1| \gg \eta^{2/3}$ a saturation sets in the nonlinear CL regime ($\lambda_{sat} \ll 1$) at the level (cf. (3.1))

$$B_{sat} \approx 0.4|\beta_1| \gg \eta^{2/3}\ .$$

Thus, here a steady nonlinear CL is realized, unlike the quasi-steady ones considered in § 2.

4. Weakly stratified mixing layer, spatial evolution (the function $\Phi_4(\lambda, Pr)$)

Consider the mixing layer treated in § 2.4, but with the weak-stratification influence also taken into account. We put, as done in § 2,

$$\xi = \varepsilon^{1/2}\mu x, \quad \Delta\omega = \mu\varepsilon^{1/2}\Omega, \quad v = \eta\varepsilon^{3/2},$$

and for the Richardson number at the critical level $y = y_c$

$$Ri \equiv Ri(y_c) = \mu\varepsilon J. \tag{4.1}$$

Here $\Omega = O(1)$, $J = O(1)$. It will also be assumed that the Prandtl number $Pr = O(1)$.

With this scaling, the linear problem properties (neutral mode + dispersion properties) are still unchanged, while nonlinear properties are now altered under the influence of stratification. Because of a regularity of the neutral mode at $y = y_c$, here one may expect the development of a quasi-steady regime of a nonlinear CL and, furthermore, a quasi-steady transition to this regime, whereas in flows with $Ri = O(1)$ the neutral mode is singular, and the disturbance grows explosively, and the nonlinear CL regime is not established evolutionarily (Churilov & Shukhman 1988)†.

Note that scaling (4.1) places reasonably stringent constraints on Ri: $Ri = O(\mu\varepsilon) = O\left(\mu\nu^{2/3}\right)$, and in the quasi-steady ($\mu \ll 1$) evolution regime of interest here this means

$$Ri \ll \nu^{2/3} \,. \tag{4.2}$$

At larger values of the Richardson number the evolution is unsteady and will be considered in a separate paper.

Equations (2.11) and (2.12) remain unaltered (in the spatial evolution problem $\partial B/\partial\tau = 0$, $\partial\Theta/\partial\tau = -\Omega$):

$$I_0 c \frac{\mathrm{d}B}{\mathrm{d}\xi} = k_0 \,⨍_{-\infty}^{\infty} \mathrm{d}Y \, \langle \zeta \sin\theta \rangle, \tag{4.3}$$

$$\left(I_0 c \frac{\mathrm{d}\Theta}{\mathrm{d}\xi} - \Omega I_1 \right) B = k_0 \,⨍_{-\infty}^{\infty} \mathrm{d}Y \, \langle \zeta \cos\theta \rangle, \tag{4.4}$$

and the dynamics of vorticity ζ inside the CL is now determined largely by the interaction with the density (temperature) disturbance P:

$$k_0 u_c' Y \frac{\partial P}{\partial \theta} + 2k_0 B \sin\theta \frac{\partial P}{\partial Y} - \frac{\eta}{Pr} \frac{\partial^2 P}{\partial Y^2} = 2k_0 B \sin\theta, \tag{4.5}$$

$$\begin{aligned} k_0 u_c' Y \frac{\partial \zeta}{\partial \theta} &+ 2k_0 B \sin\theta \frac{\partial \zeta}{\partial Y} - \eta \frac{\partial^2 \zeta}{\partial Y^2} \\ &= J k_0 \frac{\partial P}{\partial \theta} - 2 \frac{u_c'''}{u_c'} \left[c \frac{\mathrm{d}B}{\mathrm{d}\xi} \cos\theta - \left(c \frac{\mathrm{d}\Theta}{\mathrm{d}\xi} - \Omega \right) B \sin\theta \right], \end{aligned} \tag{4.6}$$

$$\frac{\partial P}{\partial Y} \to 0, \quad \frac{\partial \zeta}{\partial Y} \to 0 \quad \text{as} \quad Y \to \pm\infty \,. \tag{4.7}$$

In view of the linearity of (4.6) ζ is representable as

$$\zeta = \zeta_h + \zeta_s,$$

i.e. as the sum of the 'homogeneous' part ζ_h that is a well-known solution of (2.17) (with $\beta_1 = 0$), and the 'stratified' part ζ_s that satisfies the equation

$$k_0 u_c' Y \frac{\partial \zeta_s}{\partial \theta} + 2k_0 B \sin\theta \frac{\partial \zeta_s}{\partial Y} - \eta \frac{\partial^2 \zeta_s}{\partial Y^2} = J k_0 \frac{\partial P}{\partial \theta} \,. \tag{4.8}$$

† This is a common property of flows with a singular neutral mode (Churilov & Shukhman 1992): in a homogeneous medium the same behaviour is observed for two-dimensional disturbances when compressibility is taken into account (Goldstein & Leib 1989; Shukhman 1991) and for three-dimensional disturbances in incompressible fluids (Goldstein & Choi 1989; Wu, Lee & Cowley 1993; Churilov & Shukhman 1994).

Using the results from § 2, one may immediately write

$$\left.\begin{aligned} \int_{-\infty}^{\infty} \mathrm{d}Y \langle \zeta_h \sin\theta \rangle &= -\frac{u_c'''}{k_0 u_c'^2}\left(c\frac{\mathrm{d}\Theta}{\mathrm{d}\xi} - \Omega\right)\Phi_1(\lambda)B, \\ \int_{-\infty}^{\infty} \mathrm{d}Y \langle \zeta_h \cos\theta \rangle &= \frac{u_c'''}{k_0 u_c'^2} c \frac{\mathrm{d}B}{\mathrm{d}\xi}\Phi_2(\lambda)\,. \end{aligned}\right\} \qquad (4.9)$$

while the 'stratified' part of the problem requires separate consideration.

We now transform equation (4.8) by putting

$$\zeta_s = \tilde{\zeta}_s - \frac{J}{u_c'}\frac{\partial P}{\partial Y}\,.$$

Using (4.5) we obtain the equation

$$k_0 u_c' Y \frac{\partial \tilde{\zeta}_s}{\partial \theta} + 2k_0 B \sin\theta \frac{\partial \tilde{\zeta}_s}{\partial Y} - \eta \frac{\partial^2 \tilde{\zeta}_s}{\partial Y^2} = -\frac{Pr-1}{Pr}\eta J \frac{\partial^3 P}{\partial Y^3}, \qquad (4.10)$$

from which it is evident that $\zeta_s \propto (Pr-1)$. Since ζ_s and $\tilde{\zeta}_s$ make the same contributions to the right-hand sides of (4.3) and (4.4), it is clear that the main contribution of the stratification to the evolution equation is also proportional to $(Pr-1)$ and disappears when $Pr = 1$.

A steady nonlinear CL in terms of equations (4.5) and (4.8) was studied by Kelly & Maslowe (1970) and Haberman (1973), but their main results refer, unfortunately, to an 'uninteresting' case $Pr = 1$.

We now introduce an auxiliary function $g_4(\lambda, Pr; z, \theta)$ that satisfies the system of equations

$$\left(-\lambda\frac{\partial^2}{\partial z^2} + z\frac{\partial}{\partial\theta} + \sin\theta\frac{\partial}{\partial z}\right) g_4 = -\frac{\partial}{\partial\theta}\, g_1\left(\frac{\lambda}{Pr}; z, \theta\right), \qquad (4.11)$$

$$\left(-\frac{\lambda}{Pr}\frac{\partial^2}{\partial z^2} + z\frac{\partial}{\partial\theta} + \sin\theta\frac{\partial}{\partial z}\right) g_1\left(\frac{\lambda}{Pr}; z, \theta\right) = -2\sin\theta \qquad (4.12)$$

with the boundary conditions

$$\frac{\partial g_1}{\partial z} \to 0, \quad g_4 \to 0 \quad \text{as } z \to \pm\infty \qquad (4.13)$$

and the function

$$\Phi_4(\lambda, Pr) = \int_{-\infty}^{\infty} \langle g_4 \cos\theta \rangle \mathrm{d}z\,. \qquad (4.14)$$

Note that it follows from the symmetry properties of (4.11)–(4.13) that $\int \langle g_4 \sin\theta \rangle \mathrm{d}z = 0$. Simple calculations give

$$\left.\begin{aligned} &\int_{-\infty}^{\infty} \mathrm{d}Y \langle \zeta_s \cos\theta \rangle = J\left(\frac{B}{2u_c'^3}\right)^{1/2} \Phi_4(\lambda, Pr), \\ &\int_{-\infty}^{\infty} \mathrm{d}Y \langle \zeta_s \sin\theta \rangle = 0, \qquad \lambda = \frac{\eta (u_c')^{1/2}}{k_0 (2B)^{3/2}}. \end{aligned}\right\} \qquad (4.15)$$

The function Φ_4 is determined numerically for two values of the Prandtl number

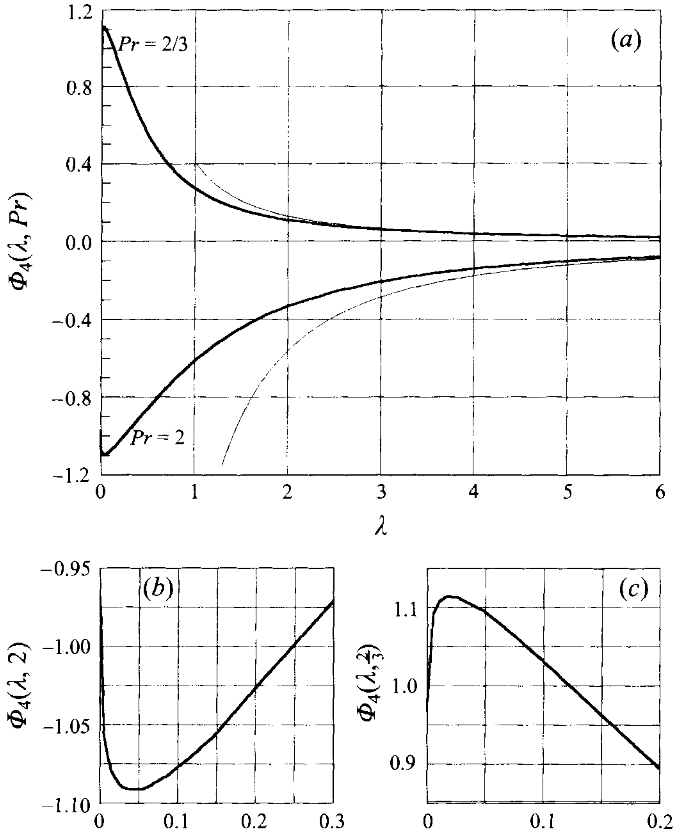

FIGURE 5. The function $\Phi_4(\lambda, Pr)$ for $Pr = 2$ and $Pr = 2/3$. Thin lines represent asymptotic expansion (4.16).

and is plotted in figure 5. In the nonlinear $\lambda \ll 1$ and viscous $\lambda \gg 1$ limits it is calculated analytically (see Appendix A):

$$\Phi_4(\lambda, Pr) = \begin{cases} C^{(4)}(Pr-1)/Pr + O(\lambda^{1/2}\ln\lambda), & \lambda \ll 1, \\ a_4(Pr)(Pr-1)\lambda^{-5/3} + O(\lambda^{-3}), & \lambda \gg 1, \end{cases} \tag{4.16}$$

where

$$C^{(4)} = \frac{3C^{(3)}}{32} - \frac{4\pi}{3} = -1.9377\ldots,$$

$$a_4(z) = -z^{5/3}\left(\frac{2}{3}\right)^{1/3}\Gamma\left(\frac{2}{3}\right)\frac{\pi}{16} \times\left[\frac{2^{8/3}}{z(z+1)^{2/3}} + F\left(\frac{4}{3},\frac{5}{3};\frac{7}{3};-\frac{z-1}{2}\right) + z^{-8/3}F\left(\frac{4}{3},\frac{5}{3};\frac{7}{3};\frac{z-1}{2z}\right)\right], \tag{4.17}$$

$F(a,b;c;z)$ being a hypergeometric function.

Upon substituting (4.9) and (4.15) into the right-hand sides of (4.3), (4.4) and solving the equations obtained for $\mathrm{d}B/\mathrm{d}\xi$ and $\mathrm{d}\Theta/\mathrm{d}\xi$, we arrive at a system of two evolution equations:

$$\frac{\mathrm{d}B}{\mathrm{d}\xi} = -\frac{2k_0^2}{c^2}\frac{u_c'''}{u_c'^2}\left[I_2\Omega B + \frac{c}{4k_0}J\left(\frac{2B}{u_c'^3}\right)^{1/2}\Phi_4(\lambda, Pr)\right]\frac{\Phi_1(\lambda)}{H(\lambda)}, \tag{4.18}$$

$$\frac{\mathrm{d}\Theta}{\mathrm{d}\xi} = \frac{\Omega}{c}\left\{1 + 2k_0^2\frac{I_0}{cH(\lambda)}\left[I_2 + \frac{cJ}{2k_0\Omega}(2Bu_c'^3)^{-1/2}\Phi_4(\lambda, Pr)\right]\right\}, \tag{4.19}$$

which give a complete solution by quadratures of the evolution problem for initially small weakly supercritical ($\gamma_L \ll \nu^{1/3}$) disturbances in a weakly stratified ($Ri \ll \nu^{2/3}$) flow. Consider the various stages and regimes of this evolution qualitatively.

Since, as has already been pointed out, $\Phi_1\Phi_2$ and H change little as λ varies from 0 to ∞, it is convenient to analyse a simplified form of (4.18):

$$\frac{\mathrm{d}B}{\mathrm{d}\xi} = \left[a\Omega + bJ\eta^{-1/3}\lambda^{1/3}\Phi_4(\lambda, Pr)\right]B\Phi_1(\lambda), \tag{4.20}$$

where a and b are positive constants of order unity. In particular, the linear growth rate is

$$\gamma_* = \pi a|\Omega|.$$

The function $\lambda^{1/3}\Phi_4(\lambda, Pr)$ goes to zero when $\lambda = 0$ and $\lambda = \infty$ (see (4.16)) and attains an extreme value (of order unity) when $\lambda = O(1)$. Therefore, as might be expected, the stratification has a marked influence upon the evolution only in the region of reasonably small supercriticality,

$$|\Omega| < J/\eta^{1/3} \ll \eta^{1/3}, \tag{4.21}$$

while at a greater supercriticality the disturbance develops in the same fashion as it does in a non-stratified flow. The sign of Φ_4 changes when $Pr = 1$; therefore, one should expect a different evolution when $Pr < 1$ and $Pr > 1$: the stratification has a stabilizing and destabilizing effect, respectively. Let us consider these cases separately.

4.1. *Flows with* $Pr < 1$ ($\Phi_4 > 0$)

In such flows all unstable disturbances, whose growth rate satisfies the inequality (4.21), are stabilized by the stratification at the level

$$B_{sat} \sim B_N = (\gamma_*\eta^{5/3}/J)^{1/2} < \eta^{2/3}, \tag{4.22}$$

i.e. still in the viscous CL regime (B_N being the nonlinearity threshold). To calculate an accurate value of B_{sat}, it is necessary to set to zero the expression between square brackets in (4.18) (or, equivalently, in (4.20)) and solve the resulting equilibrium equation for λ. The larger (corresponding to the smaller amplitude B) of its two roots, λ_1 and λ_2, should be chosen because, firstly, the evolution proceeds from $\lambda = \infty$ and, secondly, the equilibrium that corresponds to the smaller root is unstable. In the limit $|\Omega| \ll J/\eta^{1/3}$ the solution can be approximately determined analytically using (4.16):

$$B_{sat} = \left(\frac{I_2 u_c'^2}{2ck_0^{2/3}}\frac{\eta^{5/3}}{J}\left|\frac{\Omega}{a_4(Pr-1)}\right|\right)^{1/2}. \tag{4.23}$$

With increasing supercriticality, B_{sat} increases, and λ_1 and λ_2 come closer until they merge at some $\Omega = \Omega_* = O\left(J/\eta^{1/3}\right)$ (and $B_{sat} = (B_{sat})_* = O\left(\eta^{2/3}\right)$). With a larger supercriticality, the equilibrium equation has no roots, i.e. stratification cannot stop a

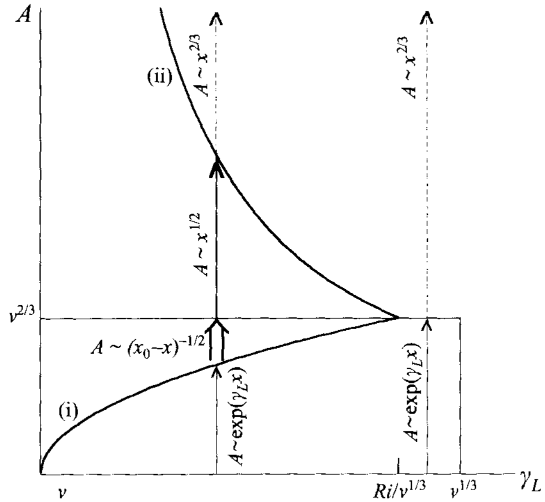

FIGURE 6. The amplitude–supercriticality diagram for a slightly stratified flow when $Ri \ll \nu^{2/3}$, $Pr > 1$. (i) $A \sim \left(\gamma_L \nu^{5/3}/Ri\right)^{1/2}$; (ii) $A \sim (Ri/\gamma_L)^2$; the arrows indicate the different evolutionary stages.

growth of disturbances, and as $B = O\left(\eta^{2/3}\right)$ they pass into the nonlinear CL regime, thereby causing the exponential growth to become power-law growth.

4.2. *Flows with* $Pr > 1$ ($\Phi_4 < 0$)

Stratification destabilizes disturbances in its region of influence (4.21), i.e. it accelerates their growth which continues indefinitely, initially in the viscous CL regime and then in the nonlinear CL regime, and can be broken down into four stages (see figure 6).

4.2.1. *Viscous CL regime*

In the limit $\lambda \gg 1$ equation (4.18) becomes

$$\frac{\mathrm{d}B}{\mathrm{d}\xi} = \left(\gamma_* B + \alpha_1 J \frac{B^2}{\eta^{5/3}}\right) B, \tag{4.24}$$

where

$$\alpha_1 = \left| \frac{4\pi a_4 (Pr - 1)}{cH(\infty)} \frac{u_c''' k_0^{8/3}}{(u_c')^{13/3}} \right| = O(1) .$$

It is evident that upon reaching the nonlinearity threshold (4.22), the initially exponential growth of the amplitude is not stopped but, on the contrary, is accelerated and becomes explosive:

$$B \propto (\eta^{5/3}/J)^{1/2}(\xi_0 - \xi)^{-1/2} . \tag{4.25}$$

Despite the increasingly speeding-up growth, the unsteady scale $l_t \sim O\left(B^2 J/\eta^{5/3}\right) \ll l_\nu$ up to $B = O\left(\eta^{2/3}\right)$ when the nonlinear scale becomes dominant and the disturbance passes into the nonlinear CL regime.

4.2.2. *Nonlinear CL regime*

In the problem at hand the transition to the nonlinear CL regime, even from the explosive growth stage, does not require for its description anything other than the evolution equation (4.18) obtained in a quasi-steady approximation, and this distinguishes it radically from similar transitions studied by Churilov & Shukhman (1995).

In the limit $\lambda \ll 1$ (4.18) gives

$$\frac{\mathrm{d}B}{\mathrm{d}\xi} = \frac{\eta (u_c')^{1/2} |C^{(1)}|}{\pi k_0 (2B)^{3/2}} \frac{H(\infty)}{H(0)} (\gamma_* B + \alpha_2 J B^{1/2}), \tag{4.26}$$

where

$$\alpha_2 = \left| \frac{\pi}{\sqrt{2}} \frac{Pr-1}{Pr} \frac{k_0 u_c'''}{c(u_c')^{7/2}} \frac{C^{(4)}}{H(\infty)} \right| = O(1) \, .$$

Immediately after the transition to the nonlinear CL regime, as long as

$$B \ll \left(\frac{J}{\gamma_*} \right)^2 , \tag{4.27}$$

the decisive role in the evolution is played by a term governed by stratification, and the amplitude increases as a power law

$$B = \alpha_3 (J \eta \xi)^{1/2}, \tag{4.28}$$

$$\alpha_3 = \left(\frac{\alpha_2}{\sqrt{2}} \frac{(u_c')^{1/2}}{\pi k_0} \frac{H(\infty)}{H(0)} \left| C^{(1)} \right| \right)^{1/2} = O(1),$$

differing from (1.5) and (2.31). And only when the amplitude has increased to the extent that the inequality (4.27) becomes the opposite, will a 'classical' power-law increase

$$B = \alpha_4 (\gamma_* \eta \xi)^{2/3}, \tag{4.29}$$

$$\alpha_4 = \left(\frac{3 (u_c')^{1/2}}{2^{5/2} \pi k_0} \frac{H(\infty)}{H(0)} \left| C^{(1)} \right| \right)^{2/3} = O(1)$$

be established.

If the supercriticality γ_* exceeds $J/\eta^{1/3}$, stratification is unimportant, and at the transition to the nonlinear CL regime an exponential growth of B becomes immediately a power-law growth as in (4.29).

5. Discussion of the results

The analysis made here shows that in the region of small supercriticality $\gamma_L \ll \nu^{1/3}$ the development of unstable disturbances in shear flows of a homogeneous and weakly stratified ($Ri \ll \nu^{2/3}$) incompressible fluid occurs at all stages in the regime of quasi-steady evolution of the vorticity ζ inside the CL. In other words, ζ adjusts itself almost instantaneously to the current value of the amplitude, and transient processes do not complicate the evolution pattern. This regime persists even at the stage of explosive growth observed in flows with $Pr > 1$.

This factor simplifies the problem considerably by reducing it generally to one evolution equation integrable by quadratures (the second equation, for the phase Θ, plays an ancillary role and is also readily integrable). It should be noted, however, that the equations obtained and their solutions in no way fit in with the formalism of 'indentation rules': the role of the CL turns out to be much more important and diverse, which manifests itself, in particular, in the diversity of the functions $\Phi_i(\lambda)$ and significant differences in their behaviour when passing from the viscous ($\lambda \gg 1$) to the nonlinear ($\lambda \ll 1$) CL regime.

Note also that the functions $\Phi_i(\lambda)$ ($i = 2, 3, 4$) cannot be assigned a pictorial meaning, as done by Haberman for his function Φ_1, and it only remains for us to treat them simply as integrals defined by the expressions (2.2), (3.3) and (4.4). Moreover, in circumstances where the CL is not a strictly steady-state one, the function Φ_1 also loses its meaning of logarithmic phase jump. It retains this meaning only in a single special case when $u_c''' = 0$ (for example, in the case of a weakly supercritical zonal flow on β-plane considered in §2.1).

The range of validity of the evolution equations obtained is significantly more extensive than the narrow framework outlined above ($\gamma_L \ll \nu^{1/3}$, $Ri \ll \nu^{2/3}$), but outside this framework their validity 'suffers a discontinuity'. Physical considerations (see the Introduction) and computer simulations show that disturbances starting from the region of an unsteady CL ($1 \gg \gamma_L \gg \nu^{1/3}$) also reach the nonlinear CL regime and, following some relaxation, their further development proceeds in a quasi-steady manner, i.e. it is described by equations (2.24), (2.25) and (4.18), (4.19) with $\lambda \ll 1$ (which permits us to confine ourselves only to corresponding asymptotic representations of the functions $\Phi_i(\lambda)$). On the other hand, at the linear stage of development the evolution is also quasi-steady: fluid particles inside the CL have virtually had no time to deviate from their unperturbed trajectories (see the Introduction). Thus, some 'range of unsteadiness' appears in the evolution pattern, which includes the relaxation process when passing from the unsteady CL regime to the nonlinear CL regime, shown in figure 1 by shading†. The 'discontinuity' in the validity of quasi-stationary equations lies in this range.

It should be emphasized that with further increasing of γ_L up to values of $O(1)$ we lose not only the possibility of a continuous quasi-steady description of evolution, but the possibility of any weakly nonlinear approach to the problem: in this case nonlinearity becomes competitive at too high amplitudes, $A = O(1)$, and weakly nonlinear theory is invalid. To overcome this difficulty of principle was not the purpose of present work.

We have carried out a detailed comparison of solutions of equations (2.24), (2.25) (to be more specific, of their particular cases, (2.36), (2.37)) with results of a numerical calculation reported by Goldstein & Hultgren (1988) in the region of parameters $\bar{\lambda} \equiv \eta / \left(|\Omega| c/2\right)^3 \gg 1$. In addition, we have compared the solution of equation (4.18) for a weakly stratified flow with the same velocity profile ($u = 1 + \tanh y$) with results of our own analogous numerical calculations (i.e. without using the quasi-steady-state approximation). All cases showed a good agreement at all stages of development of disturbances. This agreement becomes still better if allowance is made for the first correction ($O\left(\mu^{1/2}\right)$) for the unsteady vorticity in the transition (diffusion) layers between the CL and the region of the outer solution. The evolution equation in view of this correction that generalizes equations (2.24) and (4.18), has the form (we omit the derivation)

$$\frac{\mathrm{d}B}{\mathrm{d}\xi} = -\frac{2k_0^2}{c^2}\frac{u_c'''}{u_c'^2}\frac{\Phi_1(\lambda)}{H(\lambda)}\left\{ I_2 \Omega B - \frac{c^{5/2}u_c'''}{4u_c'^3 k_0}\left(\frac{\pi}{\eta}\right)^{3/2}\left(1 + \frac{u_c'^4 I_0^2}{\pi^2 u_c'''^2}\right) B \right.$$
$$\times \frac{\mathrm{d}^2}{\mathrm{d}\xi^2}\int_0^\infty \mathrm{d}\zeta\ \zeta^{-1/2} B^2(\xi - \zeta) + \frac{c}{4k_0} J \left(\frac{2B}{u_c'^3}\right)^{1/2} \Phi_4(\lambda, Pr)$$
$$\left. \times \left[1 - \frac{k_0}{u_c'}\left(\frac{cPr^3}{\pi\eta^3}\right)^{1/2}\frac{\mathrm{d}}{\mathrm{d}\xi}\int_0^\infty \mathrm{d}\zeta\ \zeta^{-1/2}\Phi_1\left(\frac{\lambda(\xi-\zeta)}{Pr}\right) B^2(\xi - \zeta)\right]\right\}. \quad (5.1)$$

† In a weakly stratified flow when $Ri \gg \nu^{2/3}$ the situation is somewhat more complicated, and a relevant problem will be considered in separate work.

The same kind of correction to the solution (only for the nonlinear CL stage, however) was obtained earlier by Goldstein & Hultgren (1988) for unstratified flow. They obtained an evolution law of the form

$$A = a_\infty(\bar\lambda\bar\xi)^{2/3}\left(1 + a_1(\bar\lambda\bar\xi)^{-1/6}\right), \quad (\bar\lambda\bar\xi)\gg 1, \tag{5.2}$$

and for the flow $u = c + \tanh y$ constants a_∞ and a_1 are

$$a_\infty = \left(\frac{3|C^{(1)}|}{2(1 + c^2C^{(1)}C^{(2)}/4)}\right)^{2/3}, \quad a_1 = \frac{2a_\infty^2}{15}\frac{\Gamma(4/3)}{\Gamma(5/6)}\left(1 + \frac{\pi^2c^2}{4}\right).$$

It is easy to see that evolution law (5.2) can be reproduced also from (5.1) with $J = 0$; however (5.1) has a larger region of applicability and permits a construction of a more exact solution for all stages of evolution, starting from $\xi = -\infty$, and not only in the developed nonlinear CL regime ($(\nu\xi) \to \infty$).

Figure 7 presents the results of calculations for a usual mixing layer and its stratified analogue. The heavy lines show the solutions of equations (2.36) and (4.18), respectively; the dashed lines correspond to the solution of the refined equation (5.1); and thin lines refer to the results of a numerical solution of 'exact' equations (for a usual mixing layer, these curves are marked by the symbol G&H). Designations in the figure are the same as in Goldstein & Hultgren (1988):

$$A \equiv \frac{2B}{(c\Omega/2)^2}, \quad \bar\xi \equiv \tfrac{1}{2}|\Omega|\xi, \quad \bar\lambda \equiv \frac{\eta}{(|\Omega|c/2)^3} \quad (c = 1),$$

when $\bar\xi \to -\infty$, $A = e^{Q\bar\xi}$, where $Q = \pi/(1 + \pi^2c^2/4)$. The stratification parameter $g \equiv J(c/4)(Pr/\eta)^{2/3}$.

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Appendix A. The limit of the nonlinear CL ($\lambda \ll 1$), and matching of solutions through the cat's-eye boundary

Solutions of the equations treated in the main text

$$\mathscr{L}g_1(\lambda; z, \theta) = -2\sin\theta, \tag{A 1a}$$

$$\mathscr{L}g_2(\lambda; z, \theta) = -2\cos\theta, \tag{A 1b}$$

$$\mathscr{L}g_3(\lambda; z, \theta) = 8\sin 2\theta, \tag{A 1c}$$

$$\mathscr{L}g_4(\lambda, Pr; z, \theta) = -\frac{\partial}{\partial\theta}\, g_1\left(\frac{\lambda}{Pr}; z, \theta\right), \tag{A 1d}$$

where

$$\mathscr{L} \equiv z\frac{\partial}{\partial\theta} + \sin\theta\frac{\partial}{\partial z} - \lambda\frac{\partial^2}{\partial z^2}$$

with boundary conditions

$$\frac{\partial g_i}{\partial z} \to 0 \ \text{as} \ z \to \pm\infty \tag{A 2}$$

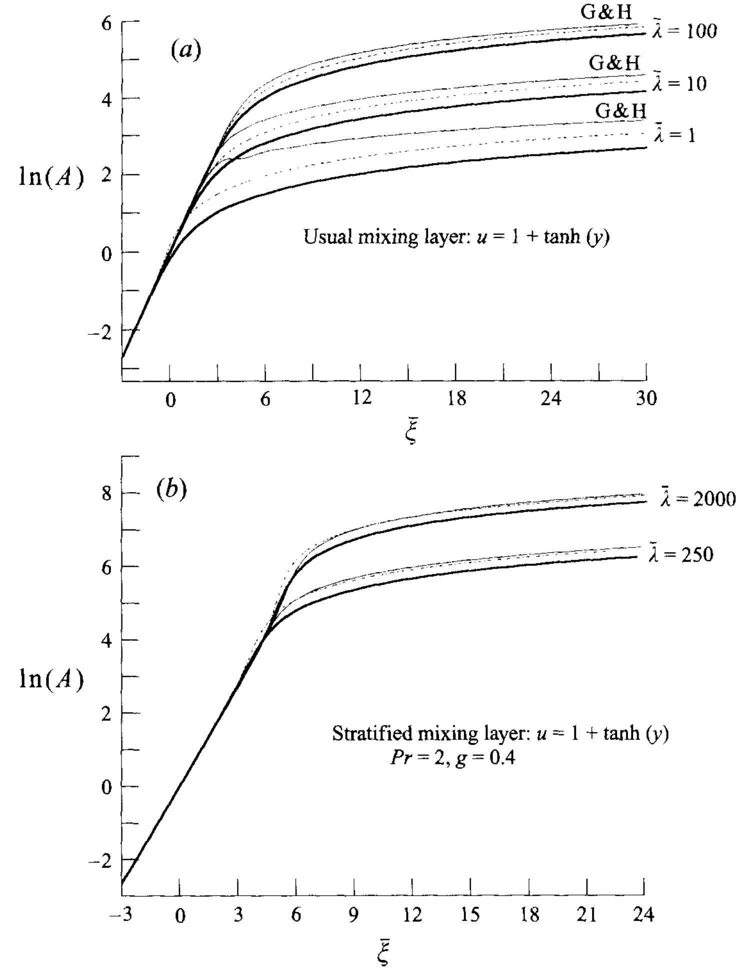

FIGURE 7. Comparison of results of computations made in the 'quasi-steady approximation' with ones based on the 'exact' numerical calculations (see text).

in the limit $\lambda \ll 1$ are reasonably straightforwardly constructed far from the separatrix $\kappa = z^2/2 + \cos\theta = 1$. There arises, however, a problem of matching the outer ($\kappa > 1$) and the inner ($|\kappa| < 1$) solutions through the region $\kappa - 1 = O(\lambda^{1/2})$ where the viscous term can no longer be considered small. Brown & Stewartson (1978) were the first to solve such a problem for (A1*a*). Following their technique we now perform a matching for the other equations (A1*b*–*d*) also.

The boundary conditions (A2) determine solutions (A1*a*–*d*) with an accuracy of up to arbitrary constants. Since the operator $\mathscr{L}$ is invariant under the transformation

$$z \to -z, \qquad \theta \to 2\pi - \theta, \tag{A 3}$$

it is convenient to choose these constants such that $g_4 \to 0$ as $z \to \pm\infty$, and g_1 and g_3 are 'odd', i.e. they change sign in the transformation (A3) (g_2 and g_4, are, obviously, 'even').

It is convenient, instead of g_i, to introduce the functions f_i :

$$g_1 = f_1 - 2z, \quad g_2 = f_2 + \frac{2}{\lambda}(\kappa - 1), \quad g_3 = f_3 + \frac{16}{3}z^3 + 16z\cos\theta, \quad g_4 = f_4 + \frac{\partial}{\partial z}\, g_1\left(\frac{\lambda}{Pr}; z, \theta\right).$$

In the variables κ, θ, equations (A1*a–d*) take the form

$$\mathscr{L}_1 f_1(\lambda;\kappa,\theta) = 0, \tag{A 4a}$$

$$\mathscr{L}_1 f_2(\lambda;\kappa,\theta) = \frac{2}{z}(1-\cos\theta) = \frac{4}{z}\sin^2\frac{\theta}{2}, \tag{A 4b}$$

$$\mathscr{L}_1 f_3(\lambda;\kappa,\theta) = 32\lambda, \tag{A 4c}$$

$$\mathscr{L}_1 f_4(\lambda, Pr;\kappa,\theta) = -\frac{Pr-1}{Pr}\lambda\frac{\partial}{\partial\kappa}\left\{z\frac{\partial}{\partial\kappa}\left[z\frac{\partial}{\partial\kappa}f_1\left(\frac{\lambda}{Pr};\kappa,\theta\right)\right]\right\}, \tag{A 4d}$$

$$\mathscr{L}_1 \equiv \frac{\partial}{\partial\theta} - \lambda\frac{\partial}{\partial\kappa}\left(z\frac{\partial}{\partial\kappa}\right), \quad z = \sigma[2(\kappa - \cos\theta)]^{1/2}, \quad \sigma = \operatorname{sgn}(z) .$$

It is also necessary to know the asymptotic representations of the inner and outer solutions of every equation. It is known that in the main order in λ

$$f_1 = \begin{cases} 4\pi\sigma\displaystyle\int_1^\kappa \frac{\mathrm{d}x}{Q(x)} + \sigma f_{10}, & \kappa > 1 \quad (f_{10} = \text{const}), \\ 0, & |\kappa| < 1, \end{cases}$$

$$f_2 = \frac{2}{\lambda}\left(1 - \kappa - \int_1^\kappa \frac{\mathrm{d}x Q_1(x)}{Q(x)}\right) + \frac{1}{\lambda}\begin{cases} f_{2e}, & \kappa > 1 \quad (f_{2e} = \text{const}), \\ f_{2i}, & |\kappa| < 1 \quad (f_{2i} = \text{const}), \end{cases}$$

$$f_3 = \begin{cases} -64\pi\sigma\displaystyle\int_1^\kappa \frac{x\,\mathrm{d}x}{Q(x)} + \sigma f_{30}, & \kappa > 1 \quad (f_{30} = \text{const}), \\ 0, & |\kappa| < 1, \end{cases}$$

$$f_4 = \frac{Pr-1}{Pr}\begin{cases} 1 - \dfrac{8\pi^2\kappa}{Q^2(\kappa)}, & \kappa > 1, \\ f_{40} = \text{const}, & |\kappa| < 1 \end{cases}$$

(for f_4 see also Kelley & Maslowe 1970). Here

$$Q(\kappa) = \int_{-\theta_m}^{\theta_m} \mathrm{d}\theta[2(\kappa + \cos\theta)]^{1/2}, \quad Q_1(\kappa) = \int_{-\theta_m}^{\theta_m} \mathrm{d}\theta\cos\theta[2(\kappa+\cos\theta)]^{1/2},$$

$$\theta_m = \begin{cases} \arccos(-\kappa), & |\kappa| < 1, \\ \pi, & \kappa \geqslant 1 \end{cases}$$

or, in terms of the complete elliptic integrals E and K:

$$Q = \begin{cases} \dfrac{8}{q}\mathrm{E}(q), & q^2 = \dfrac{2}{1+\kappa}, \quad \kappa > 1, \\ 8[\mathrm{E}(q) - (1-q^2)\mathrm{K}(q)], & q^2 = \dfrac{1+\kappa}{2}, \quad |\kappa| < 1, \end{cases}$$

$$Q_1 = \begin{cases} \dfrac{8}{3q}[\kappa \mathrm{E}(q) - (\kappa - 1)\mathrm{K}(q)], & q^2 = \dfrac{2}{1+\kappa}, \quad \kappa > 1, \\ \dfrac{8}{3}[\kappa \mathrm{E}(q) + (1-\kappa)\mathrm{K}(q)/2], & q^2 = \dfrac{1+\kappa}{2}, \quad |\kappa| < 1. \end{cases}$$

As $\kappa \to 1$ ($\kappa - 1 = \lambda^{1/2} s$):

$$f_1 = \begin{cases} \sigma \left[f_{10} + \dfrac{\pi}{2}\lambda^{1/2} s + O(\lambda s^2 \ln |s|) \right], & \kappa > 1, \\ 0, & |\kappa| < 1, \end{cases} \tag{A 5a}$$

$$f_2 = \frac{1}{\lambda} \begin{cases} f_{2e} - \frac{8}{3}\lambda^{1/2} s + O(\lambda s^2 \ln |s|), & \kappa > 1, \\ f_{2i} - \frac{8}{3}\lambda^{1/2} s + O(\lambda s^2 \ln |s|), & |\kappa| < 1, \end{cases} \tag{A 5b}$$

$$f_3 = \begin{cases} \sigma \left[f_{30} - 8\pi \lambda^{1/2} s + O(\lambda s^2 \ln |s|) \right], & \kappa > 1, \\ 0, & |\kappa| < 1, \end{cases} \tag{A 5c}$$

$$f_4 = \frac{Pr - 1}{Pr} \begin{cases} 1 - \dfrac{\pi^2}{8} + O(\lambda^{1/2} s \ln |s|), & \kappa > 1, \\ f_{40}, & |\kappa| < 1. \end{cases} \tag{A 5d}$$

In the variables θ and $s = \lambda^{-1/2}(\kappa - 1)$ (A4) become

$$\mathscr{L}_2 f_1(\lambda; s, \theta) \equiv \left(\frac{\partial}{\partial \theta} - \sigma \frac{\partial}{\partial s} \left(2\lambda^{1/2} s + 4 \sin^2 \tfrac{1}{2}\theta\right)^{1/2} \frac{\partial}{\partial s} \right) f_1 = 0, \tag{A 6a}$$

$$\mathscr{L}_2 f_2(\lambda; s, \theta) = \frac{4\sigma \sin^2 \frac{1}{2}\theta}{\left(2\lambda^{1/2} s + 4 \sin^2 \frac{1}{2}\theta\right)^{1/2}}, \tag{A 6b}$$

$$\mathscr{L}_2 f_3(\lambda; s, \theta) = 32\lambda, \tag{A 6c}$$

$$\mathscr{L}_2 f_4(\lambda, Pr; s, \theta) = \frac{Pr - 1}{Pr} \lambda^{-1/2} \times \frac{\partial}{\partial s} \left\{ \left(2\lambda^{1/2} s + 4 \sin^2 \tfrac{1}{2}\theta\right)^{1/2} \frac{\partial}{\partial s} \left[\left(2\lambda^{1/2} s + 4 \sin^2 \tfrac{1}{2}\theta\right)^{1/2} \frac{\partial}{\partial s} f_1 \left(\frac{\lambda}{Pr}; s, \theta \right) \right] \right\}. \tag{A 6d}$$

Following Brown & Stewartson (1978) we construct solutions of (A6*a–d*) by the successive approximation method. Far from the cat's-eye corners (when $\sin \theta/2 \gg \lambda^{1/4}$)

$$\left(2\lambda^{1/2} s + 4 \sin^2 \tfrac{1}{2}\theta\right)^{1/2} = 2 \sin \tfrac{1}{2}\theta + O\left(\frac{\lambda^{1/2}}{\sin \frac{1}{2}\theta} \right) = 4 \sin \tfrac{1}{4}\theta \cos \tfrac{1}{4}\theta + O\left(\frac{\lambda^{1/2}}{\sin \frac{1}{2}\theta} \right), \tag{A 7}$$

and it is convenient to introduce the variable

$$\tau = \begin{cases} 8\sin^2 \frac{1}{4}\theta, & \sigma > 0, \\ 8\cos^2 \frac{1}{4}\theta, & \sigma < 0, \end{cases}$$

which in the corners takes the value 0 and 8. In this case the operator $\mathscr{L}_2$ is transformed into the diffusion operator.

(*a*) *Calculation of* f_1

Equation (A6*a*) at the main order becomes

$$\frac{\partial f_1}{\partial \tau} - \frac{\partial^2 f_1}{\partial s^2} = 0 . \tag{A 8}$$

The function f_1 is odd with respect to (A3) and is 2π-periodic in θ, therefore

$$f_1(s,0) = f_1(s,8)\ \mathrm{sgn}\, s \tag{A 9}$$

and it is conveniently represented in terms of the functions $f(s)$ and $g(s)$ such that

$$\left.\begin{aligned} &f(s) = g(s) = 0, \quad \text{when} \quad s < 0, \\ &f_1(s,8) = f(s) + g(-s); \qquad f_1(s,0) = f(s) - g(-s) . \end{aligned}\right\} \tag{A 10}$$

Upon pasing to the Fourier transform,

$$F(s) = \int_{-\infty}^{\infty} \mathrm{d}k\, F_k \exp(\mathrm{i}ks),$$

one can see that

$$f_{1k} = f_{1k}(0) \exp(-k^2\tau) .$$

The functions f_k and g_k are analytic in the lower half-plane ($\mathrm{Im}k < 0$) and, in view of (A10), satisfy the equation

$$g_{-k} = -f_k \tanh(4k^2).$$

By solving it using the Wiener–Hopf method (for details see Brown & Stewartson 1978), we get

$$f_k = -\frac{B}{k^2 F_-(k)}, \quad g_{-k} = BF_+(k), \quad B = \text{const},$$

$$\left.\begin{aligned} \ln F_\pm(k) &= \tfrac{1}{2} \ln \frac{\tanh(4k^2)}{k^2} \pm \mathrm{i}I(k), \\ I(k) &= \frac{8}{\pi} \int_0^{\infty} \frac{q\, \mathrm{d}q}{\sinh(8q^2)} \ln\left|\frac{q-k}{q+k}\right| + \frac{\pi}{2}\, \mathrm{sgn}\, k . \end{aligned}\right\} \tag{A 11}$$

Since $I(-k) = -I(k)$ and $F_+(-k) = F_-(k)$, we obtain

$$\left.\begin{aligned} f(s) &= -B \int_{-\infty}^{\infty} \frac{\mathrm{d}k\ \exp(\mathrm{i}ks)}{k^2 F_-(k)}, \\ g(s) &= B \int_{-\infty}^{\infty} \mathrm{d}k\ \exp(\mathrm{i}ks) F_+(-k) = B \int_{-\infty}^{\infty} \mathrm{d}k\ \exp(\mathrm{i}ks) F_-(k), \end{aligned}\right\} \tag{A 12}$$

where the integration domain is the real axis with indentation of the pole $k = 0$ in the first integral from below. Integrals in (A12) can be evaluated by invoking the residue theorem, taking into consideration that in the upper half-plane zeros and poles of $F_{-}(k)$ coincide with those of $k^2 \tanh(4k^2)$ which are determined by

$$4k^2 = \mathrm{i}\pi n \quad \text{and} \quad 4k^2 = \mathrm{i}\pi m - \mathrm{i}\pi/2$$

respectively. Residues in the poles of F_{-} will give a series of rapidly decreasing (with increasing s) exponents; therefore

$$g(s) = O\left(\exp\left[-\tfrac{1}{4}\pi^{1/2}s\right]\right), \quad s \to +\infty. \tag{A 13}$$

In $f(s)$ the main contribution is made by the residue at the point $k = 0$. We get

$$f(s) = \pi B\left[s - \frac{4}{\pi^{1/2}}\left(\sqrt{2}-1\right)\zeta\left(\tfrac{1}{2}\right)\right] + O\left(\exp\left[-\left(\tfrac{1}{8}\pi\right)^{1/2}s\right]\right), \quad s \to +\infty, \tag{A 14}$$

where $\zeta(z)$ is the Riemann zeta function.

Matching to (A5*a*) gives

$$B = \tfrac{1}{2}\sigma\lambda^{1/2}, \quad f_{10} = -2(\pi\lambda)^{1/2}\left(\sqrt{2}-1\right)\zeta\left(\tfrac{1}{2}\right) \approx 2.1438\lambda^{1/2}; \tag{A 15}$$

thus, the function f_1, as shown for the first time by Brown & Stewartson (1978), undergoes on the cat's-eye boundary a 'jump' of $O\left(\lambda^{1/2}\right)$ which determines the expansion term (following the main term) of Φ_1 when $\lambda \ll 1$:

$$\Phi_1(\lambda) = C^{(1)}\lambda + D^{(1)}\lambda^{3/2} + \cdots, \quad D^{(1)} = -4\pi^{1/2}\left(\sqrt{2}-1\right)\zeta\left(\tfrac{1}{2}\right) = 4.2876\ldots \tag{A 16}$$

(*b*) *Calculation of* f_3

It follows from (A5*a*) that $f_3 \geqslant O\left(\lambda^{1/2}\right)$, hence not only is f_3 odd with respect to (A3) but it also satisfies, in a first approximation, the same equation (A8) far from the cat's-eye corners. Therefore, f_3 differs from f_1 only by the choice of the constant B in (A11)–(A14). In particular, as $s \to +\infty$

$$f_3 = \pi B_3\left[s - \frac{4}{\pi^{1/2}}\left(\sqrt{2}-1\right)\zeta\left(\tfrac{1}{2}\right)\right] + O\left(\exp\left[-\left(\tfrac{1}{8}\pi\right)^{1/2}s\right]\right)$$

and matching to (A5*c*) gives

$$B_3 = -8\sigma\lambda^{1/2}, \quad f_{30} = 32(\pi\lambda)^{1/2}\left(\sqrt{2}-1\right)\zeta\left(\tfrac{1}{2}\right) \approx -34.3006\lambda^{1/2}.$$

The function f_3 also undergoes a 'jump' $O\left(\lambda^{1/2}\right)$ on the cat's-eye boundary, and

$$\Phi_3(\lambda) = C^{(3)}\lambda + D^{(3)}\lambda^{3/2} + \cdots, \quad D^{(3)} = 64\pi^{1/2}\left(\sqrt{2}-1\right)\zeta\left(\tfrac{1}{2}\right) = -68.6012\ldots \tag{A 17}$$

(*c*) *Calculation of* f_2

In accordance with (A5*b*), we seek the solution in the form of an expansion

$$f_2 = \lambda^{-1}f_2^{(-1)} + \lambda^{-1/2}f_2^{(-1/2)} + \cdots.$$

The equation for $f_2^{(-1)}$ is homogeneous and coincides, far from the cat's-eye corners, with (A8), but – unlike f_1 – the function f_2 is even with respect to (A3), and

$$f_2(s, 0) = f_2(s, 8)$$

at all s. In Fourier transform, we obtain the equation

$$f_{2k}^{(-1)}[1 - \exp(-8k^2)] = 0, \quad \text{i.e.} \quad f_{2k}^{(-1)} = a\delta(k) + \mathrm{i}b\delta'(k),$$

where δ and δ' are, respectively, the Dirac delta-function and its derivative. Upon inverse Fourier transforming, we obtain the equation

$$f_2^{(-1)} = a + bs \, .$$

Matching to (A5b) gives

$$f_{2e} = f_{2i} = a, \quad b = 0.$$

It must be emphasized that the constant is an exact solution of the equation $\mathscr{L}_2 F = 0$, and taking into account the cat's-eye corners does not make any additional contributions to the following orders of the perturbation theory.

The iteration $f_2^{(-1/2)}$ also satisfies (A8) far from the corners, and in view of (A5b) we obtain

$$f_2^{(-1/2)} = -2s/3 \, .$$

Thus, to an accuracy of $O\left(\lambda^{-1/2}\right)$ the function f_2 is continuous on the cat's-eye boundary.

(d) Calculation of f_4

It follows from (A11) and (A15) that in the region considered (the vicinity of the separatrix $\kappa = 1$) $f_1 = O\left(\lambda^{1/2}\right)$; therefore, put $f_1 = \lambda^{1/2} h$. The function h is a simple modification of f_1: it is necessary merely to everywhere change k for $k/Pr^{1/2}$ and put $B = \sigma/(2Pr)$.

Far from the cat's-eye corners, equation (A6d) becomes

$$\frac{\partial f_4}{\partial \tau} = \frac{\partial^2 f_4}{\partial s^2} + \frac{\sigma(Pr-1)}{2Pr}[\tau(8-\tau)]^{1/2} \frac{\partial^3 h}{\partial s^3},$$

or, in Fourier-transform,

$$\frac{\mathrm{d} f_{4k}}{\mathrm{d}\tau} = -k^2 f_{4k} - \frac{\mathrm{i}\sigma}{2}\frac{Pr-1}{Pr} k^3 [\tau(8-\tau)]^{1/2} h_k(\tau), \qquad h_k(\tau) = h_k(0)\exp\left(-\frac{k^2}{Pr}\tau\right) \, .$$

Its solution is

$$f_{4k}(\tau) = f_{4k}(0)\exp(-k^2\tau) - \frac{\mathrm{i}\sigma}{2}\left(1 - \frac{1}{Pr}\right) k^3 h_k(0) \times \int_0^\tau \mathrm{d}t [t(8-t)]^{1/2} \exp\left[-k^2\left(\tau - \frac{Pr-1}{Pr}t\right)\right] \, . \tag{A 18}$$

The function f_4 is even with respect to (A3); hence

$$f_4(s,0) = f_4(s,8) \quad \text{and} \quad f_{4k}(0) = f_{4k}(8),$$

which in view of (A18) leads to the equation

$$\left(1 - \exp(-8k^2)\right) f_{4k}(0) = -4\pi \mathrm{i} k^3 \sigma h_k(8) \frac{Pr-1}{Pr} \Phi\left(\tfrac{3}{2}, 3; 8k^2(1-Pr)/Pr\right), \tag{A 19}$$

where $\Phi(a, c; z)$ is a Kummer's confluent hypergeomertric function.

Upon substituting in (A11) k for $k/Pr^{1/2}$ and putting $B = \sigma/(2Pr)$, from (A19) we obtain

$$f_{4k}(0) = \frac{Pr-1}{Pr} \frac{4\mathrm{i}\pi k \Phi\left(\tfrac{3}{2}, 3; 8k^2(1-Pr)/Pr\right)}{\left[1 - \exp(-8k^2)\right]\left[1 + \exp(8k^2/Pr)\right] F_-\left(k/Pr^{1/2}\right)} + a_1\delta(k) + \mathrm{i} b_1 \delta'(k),$$

and after an inverse Fourier transform

$$f_4(s,0) = 4i\pi\frac{Pr-1}{Pr}\int_{-\infty}^{\infty}\mathrm{d}k\,\frac{k\exp(iks)\Phi\left(\frac{3}{2},3;8k^2(1-Pr)/Pr\right)}{\left[1-\exp(-8k^2)\right]\left[1+\exp(8k^2/Pr)\right]F_-\left(k/Pr^{1/2}\right)} + a_1 + b_1 s,$$

where integration, as in (A13), proceeds along the real axis with indentation of the pole $k=0$ from below. When $|s|\to\infty$ we obtain

$$f_4(s,0) = \begin{cases} a_1 + b_1 s - \dfrac{Pr-1}{Pr}\dfrac{\pi^2}{4} + O\left(\exp\left[\left(\frac{1}{8}\pi\right)^{1/2}s\right] + \exp\left[-\left(\frac{1}{8}\pi Pr\right)^{1/2}s\right]\right), & s\to+\infty \\ a_1 + b_1 s + O\left(\exp\left[\left(\frac{1}{8}\pi\right)^{1/2}s\right] + \exp\left[-\frac{1}{4}(\pi Pr)^{1/2}s\right]\right), & s\to-\infty \end{cases} \tag{A 20}$$

Matching to (A5d) gives

$$f_{40} = a_1 = \frac{Pr-1}{Pr}\left(1+\frac{\pi^2}{8}\right), \qquad b_1 = 0\,.$$

Thus, f_4 undergoes on the cat's-eye boundary an $O(1)$ 'jump' caused by a jump of the derivative of the function f_1. This relation is revealed in the most clear way by using a generalized Prandtl–Batchelor theorem (see, for example, Goldstein & Hultgren 1988; Churilov & Shukhman 1995). In order to obtain it, we introduce a function $G(z,\theta)$ such that

$$\frac{\partial G}{\partial z} = z f_4,$$

rewrite (A4d) in z and θ variables and integrate it once over z:

$$\frac{\partial G}{\partial \theta} + f_4\sin\theta = \lambda\frac{\partial f_4}{\partial z} + \frac{Pr-1}{Pr}\lambda\frac{\partial^2}{\partial z^2}\,g_1\left(\frac{\lambda}{Pr};z,\theta\right) + g_0(\theta)\,.$$

Since $f_4 = \dfrac{1}{z}\dfrac{\partial G}{\partial z}$, we obtain

$$\left(\frac{\partial G}{\partial \theta}\right)_\kappa = \lambda\frac{\partial g_4}{\partial z} + \frac{Pr-1}{Pr}\lambda\frac{\partial^2}{\partial z^2}\,g_1\left(\frac{\lambda}{Pr};z,\theta\right) + g_0(\theta)\,.$$

Integrating over any closed contour $\kappa = \text{const}$ (when $\kappa > 1$ the contour is closed by segments $\theta=0$ and $\theta=2\pi$, see Churilov & Shukhman 1995), we obtain

$$\oint \mathrm{d}\theta\left[\frac{\partial f_4}{\partial z} + \frac{Pr-1}{Pr}\frac{\partial^2}{\partial z^2}\,g_1\left(\frac{\lambda}{Pr};z,\theta\right)\right] = 0\,.$$

Since f_4 is even and g_1 is odd with respect to (A3), the integrand is odd, and the integral over the upper arc of the contour equals the integral over the lower arc, i.e.

$$\int_{\pi-\theta_m}^{\pi+\theta_m}\mathrm{d}\theta\left[\frac{\partial f_4}{\partial z} + \frac{Pr-1}{Pr}\frac{\partial^2}{\partial z^2}\,g_1\left(\frac{\lambda}{Pr};z,\theta\right)\right] = 0$$

at any κ and λ. In the limit $\lambda\to 0$ we choose $\kappa_1 = 1-\epsilon$, $\kappa_2 = 1+\epsilon$ where $\epsilon = O\left(\lambda^{1/2}\right)$

and integrate this equality over κ from κ_1 to κ_2. The jump of f_4 on the cat's-eye boundary is

$$[f_4]Q(1) = -4\pi\sigma\frac{Pr-1}{Pr}\left[\frac{\mathrm{d}g_1}{\mathrm{d}\kappa}\right] + O\left(\lambda^{1/2}\right) .$$

Since, according to (A5a) $\left[\mathrm{d}g_1/\mathrm{d}\kappa\right] = 4\pi\sigma/Q(1) + O\left(\lambda^{1/2}\right)$ and $Q(1) = 8$, hence

$$[f_4] = -\frac{Pr-1}{Pr}\frac{16\pi^2}{Q^2(1)} + O\left(\lambda^{1/2}\right) = -\frac{Pr-1}{Pr}\frac{\pi^2}{4} + O\left(\lambda^{1/2}\right),$$

in complete accordance with (A20).

Appendix B. Algorithm for numerical solution of equations for $g_1, \ldots, g_4$ and calculation of the functions $\Phi_1(\lambda), \ldots, \Phi_4(\lambda, Pr)$

We now expand g_i in terms of harmonics:

$$g_i(\lambda; z, \theta) = \sum_{n=-\infty}^{\infty} f_n^{(i)}(\lambda; z)\mathrm{e}^{in\theta}, \qquad f_n^{(i)} = \overline{f_{-n}^{(i)}} . \tag{B 1}$$

For $i = 1, 2, 3$ we obtain:

$$\mathrm{i}zf_1^{(i)} + \tfrac{1}{2}\mathrm{i}f_2^{(i)\prime} + \frac{1}{4\lambda}\left(f_1^{(i)} - \overline{f_1^{(i)}}\right) - \lambda f_1^{(i)\prime\prime} = r_1^{(i)}, \tag{B 2}$$

$$\mathrm{i}nzf_n^{(i)} + \tfrac{1}{2}\mathrm{i}\left({f_{n+1}^{(i)}}' - {f_{n-1}^{(i)}}'\right) - \lambda f_n^{(i)\prime\prime} = r_n^{(i)}, \quad n \geqslant 2 . \tag{B 3}$$

The zeroth harmonic is eliminated from the equations by means of the relationship

$$\lambda f_0^{(i)\prime} = \tfrac{1}{2}\mathrm{i}\left(f_1^{(i)} - \overline{f_1^{(i)}}\right) . \tag{B 4}$$

Here

$$r_n^{(1)} = \mathrm{i}\delta_{1n}, \qquad r_n^{(2)} = -\delta_{1n}, \qquad r_n^{(3)} = -4\mathrm{i}\delta_{2n} . \tag{B 5}$$

For $i = 4$ we have

$$\mathrm{i}zf_1^{(4)} + \tfrac{1}{2}\mathrm{i}f_2^{(4)\prime} + \frac{1}{4\lambda}\left(f_1^{(4)} - \overline{f_1^{(4)}}\right) - \lambda f_1^{(4)\prime\prime} = -\mathrm{i}\widetilde{f_1^{(1)}}, \tag{B 6}$$

$$\mathrm{i}nzf_n^{(4)} + \tfrac{1}{2}\mathrm{i}\left({f_{n+1}^{(4)}}' - {f_{n-1}^{(4)}}'\right) - \lambda f_n^{(4)\prime\prime} = -\mathrm{i}n\widetilde{f_n^{(1)}}, \quad n \geqslant 2, \tag{B 7}$$

where

$$\widetilde{f_n^{(1)}} = f_n^{(1)}\left(\frac{\lambda}{Pr}; z\right) . \tag{B 8}$$

Boundary conditions are formulated by specifying asymptotic expansions as $z \to \pm\infty$:

$$f_1^{(1)} = \frac{1}{z} + \frac{5}{8z^5} - \frac{2\mathrm{i}\lambda}{z^4}, \quad f_n^{(1)} = \frac{a_n}{z^{2n-1}}, \quad a_{n+1} = -\frac{2n-1}{2(n+1)}a_n; \quad a_1 = 1, \tag{B 9}$$

$$f_1^{(2)} = \frac{\mathrm{i}}{z} - \frac{1}{2\lambda z^2}, \quad f_n^{(2)} = \frac{\mathrm{i}a_n}{z^{2n-1}} + \frac{b_n}{2\lambda z^{2n}}, \quad b_n = \frac{(-1)^n}{n!}, \tag{B 10}$$

$$f_1^{(3)} = -\frac{1}{z^3} + \frac{16i\lambda}{z^6}, \quad f_2^{(3)} = -\frac{2}{z} + \frac{2i\lambda}{z^4}, \quad f_n^{(3)} = \frac{c_n}{z^{2n-3}}, \quad c_{n+1} = -\frac{2n-3}{2(n+1)}c_n; \quad c_2 = -2, \tag{B 11}$$

$$f_n^{(4)} = -\frac{d_n}{z^{2n}}, \quad d_n = e_n a_n, \quad e_{n+1} = 1 + \frac{2n}{2n-1}e_n; \quad e_1 = 1\,. \tag{B 12}$$

The region of numerical calculation is the interval $[-z_m, z_m]$, outside which contributions to integrals of our interest are calculated using (B9)–(B12). We obtain

$$\Phi_1(\lambda) = \int_{-z_m}^{z_m} \mathrm{d}z\, \mathrm{Im}\left(f_1^{(1)}\right) + \frac{4\lambda}{3z_m^3}, \tag{B 13}$$

$$\Phi_2(\lambda) = \int_{-z_m}^{z_m} \mathrm{d}z\, \mathrm{Re}\left(f_1^{(2)}\right) - \frac{1}{\lambda z_m} + \frac{4\lambda}{3z_m^3}, \tag{B 14}$$

$$\Phi_3(\lambda) = \int_{-z_m}^{z_m} \mathrm{d}z\, \mathrm{Im}\left(f_1^{(3)}\right) - \frac{32\lambda}{5z_m^5}, \tag{B 15}$$

$$\Phi_4(\lambda, Pr) = \int_{-z_m}^{z_m} \mathrm{d}z\, \mathrm{Re}\left(f_1^{(4)}\right) - \frac{2}{z_m} + O\left(z_m^{-5}\right). \tag{B 16}$$

Solutions of stationary equations (B2), (B3), (B6), (B7) were determined by the method of establishment, i.e. instead of them we solved unsteady equations for the functions $f_n^{(i)}(\lambda; z, t)$:

$$\frac{\partial f_1^{(i)}}{\partial t} + \mathrm{i}z f_1^{(i)} + \frac{\mathrm{i}}{2}A f_2^{(i)\prime} + \frac{A^2}{4\lambda}\left(f_1^{(i)} - \overline{f_1^{(i)}}\right) - \lambda f_1^{(i)\prime\prime} = r_1^{(i)}, \tag{B 17}$$

$$\frac{\partial f_n^{(i)}}{\partial t} + \mathrm{i}nz f_n^{(i)} + \frac{\mathrm{i}}{2}A\left(f_{n+1}^{(i)}{}' - f_{n-1}^{(i)}{}'\right) - \lambda f_n^{(i)\prime\prime} = r_n^{(i)}, \quad n \geqslant 2\,, \tag{B 18}$$

and in much the same way for $f_n^{(4)}$.

Let the 'amplitude' A be a specified function of time t which is adiabatically turned on when $t = 0$ and smoothly goes to unity when $t = \tau$. As a certain time T ($T \leqslant 10\tau$) elapses, the required stationary solutions are established. To represent $A(t)$, we take the function

$$A(t) = \frac{t}{\tau}\left[1 - \left(1 - \frac{t}{\tau}\right)^2\right] \tag{B 19}$$

and choose $\tau = \lambda^{-1}$ when $\lambda < 1$ and $\tau = 1$ when $\lambda > 1$. When $t = 0$ (i.e. when $A = 0$) the initial functions $f_n^{(i)}$ were specified as solutions of corresponding linear equations:

$$f_n^{(1)}(\lambda; z, 0) = \frac{1}{\lambda^{1/3}}F\left(\frac{z}{\lambda^{1/3}}\right)\delta_{n1}, \tag{B 20}$$

$$f_n^{(2)}(\lambda; z, 0) = \frac{\mathrm{i}}{\lambda^{1/3}}F\left(\frac{z}{\lambda^{1/3}}\right)\delta_{n1}, \tag{B 21}$$

$$f_n^{(3)}(\lambda; z, 0) = \frac{2^{4/3}}{\lambda^{1/3}}F\left(\frac{2^{1/3}z}{\lambda^{1/3}}\right)\delta_{n2}, \tag{B 22}$$

$$f_n^{(4)}(\lambda, Pr; z, 0) = \left(\frac{Pr}{\lambda}\right)^{2/3} G\left[z\left(\frac{Pr}{\lambda}\right)^{1/3}; Pr\right]\delta_{n1}. \tag{B 23}$$

Here

$$F(z) = \mathrm{i}\int_0^\infty \mathrm{d}t\, \exp(-\tfrac{1}{3}t^3 - \mathrm{i}tz), \tag{B 24}$$

$$G(z, Pr) = \int_0^\infty \mathrm{d}t\, t \exp(-\mathrm{i}tz) \int_0^1 \mathrm{d}v\, v^3 \exp\left\{-\tfrac{1}{3}t^3\left[Pr - (Pr-1)v^3\right]\right\}. \tag{B 25}$$

The boundary conditions (B9)–(B12) that are specified at $z = \pm z_m$ must also be redefined for $A \neq 1$:

$$f_1^{(1)} = \frac{1}{z} + \frac{5A^2}{8z^5} - \frac{2\mathrm{i}\lambda}{z^4}, \quad f_n^{(1)} = \frac{a_n}{z^{2n-1}}, \quad a_{n+1} = -\frac{2n-1}{2(n+1)}a_n A; \quad a_1 = 1, \tag{B 26}$$

$$f_1^{(2)} = \frac{\mathrm{i}}{z} - \frac{A^2}{2\lambda z^2} - \frac{\mathrm{i}}{\lambda z^3}A\frac{\mathrm{d}A}{\mathrm{d}t}, \quad f_n^{(2)} = \frac{\mathrm{i}a_n}{z^{2n-1}} + \frac{b_n A^{n+1}}{2\lambda z^{2n}}, \quad b_n = \frac{(-1)^n}{n!}, \tag{B 27}$$

$$f_1^{(3)} = -\frac{A}{z^3} + \frac{16\mathrm{i}\lambda}{z^6}A, \quad f_2^{(3)} = -\frac{2}{z} + \frac{2\mathrm{i}\lambda}{z^4}, \quad f_n^{(3)} = \frac{c_n}{z^{2n-3}}, \tag{B 28}$$
$$c_{n+1} = -\frac{2n-3}{2(n+1)}c_n A; \quad c_2 = -2$$

(the boundary condition (B12) for $f^{(4)}$ retains its form, except for the re-defining of a_n (see (B26)).

The system of partial diferential equations was solved by a predictor–corrector method similar to that decribed by Goldstein & Hultgren (1988).

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